PRELIMINARY

NUMERICAL INTEGRATION OF MOTION IN SPECIAL RELATIVITY

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This article applies numerical integration of ordinary differential equations to solve the dynamics of body motion in special relativity. The method discussed parallels integration of body motion in Newtonian mechanics closely using numerical techniques such as Runge-Kutta or Predictor-Corrector. The end results of the integration are multi-dimensional trajectories and velocities of particles or rigid bodies undergoing applied forces with respect to time. Only translations of motion are addressed; rotational motion is not considered. Transformation equations are given in vector form to streamline computer computations. Simplified dynamics problems in special relativity are solved analytically, and results are compared to numerical integration of the same problems.

INTRODUCTION

Computer analyses have been abundantly applied in integrating ordinary differential equations of motion in Newtonian physics. These analyses typically apply numerical integration methods such as Runge-Kutta or Predictor-Corrector. Trajectories of motion have been modeled on computer for many moving objects such as rockets, satellites, airplanes, and gun projectiles. The same methods can be applied to the ordinary differential equations of motion in special relativity. This article outlines a procedure which applies numerical integration to equations of motion in special relativity. The article addresses multi-dimensional translations of rigid body or particle motion with respect to time and distance. Rotational aspects of body motion are not considered.

Several topics are discussed within the context of special relativity which may be useful in computing body motion numerically. First, a summary is given describing general methods of numerical integration of coupled ordinary differential equations. Then, a method is outlined for numerical integration of body motion in special relativity. The method defines variables of integration, and draws parallels between Newtonian and special relativity applications. Vector equations are given to transform several quantities between inertial reference frames, if the need arises. In vector form, these equations decrease computations and code size. Force definitions must meet the stringent requirements imposed by propagation and transformation effects in special relativity, and two force equations are derived for application in numerical analyses. Lastly, some simplified analytical cases of body motion are solved algebraically, and analytic results are compared with numerical calculations using the methods outlined herein.

I. SUMMARY OF NUMERICAL METHODS FOR INTEGRATION

Methods such as Runge-Kutta are able to integrate equations of motion in special relativity because of the wide latitude the methods give in expressing the definitions of the derivatives being integrated. The problem of solving ordinary differential equations is reduced to the study of a set of N coupled first order differential equations[1]. The differential equations are each derivatives of a function g_i, ($i = 1,2,...,N$) having the general form

$$\frac{d}{dt} g_i(t) = f_i(t, g_1, \ldots, g_N) \tag{1}$$

The conditions on the derivative functions f_i are that they be known, be functions of the independent integration parameter t, and be functions of the dependent integration functions g_i. All derivatives are taken with respect to the single integration parameter t. The functions g_i are calculated at the instantaneous value of the parameter t. Other than these criteria, the functions can take on any form the user requires.

In applying Eq. (1) to equations of motion, variables such as location, velocity, momentum, and mass become the dependent integration functions g_i which are integrated by the integration algorithm. Variables such as velocity, acceleration, force, and mass time rate of change become the derivative functions f_i identified by the user. Time is the independent integration parameter t. In coding the integration on a computer, an array of variables g_i with size N is given initial values. This array is then passed to an integration algorithm along with a derivative function call. This function defines a derivative value f_i for each variable g_i in the array for each instant in time. The integration algorithm steps through a given time interval, and the equations of motion are integrated. Source code is given for function calls of several examples of motion integration in Sec. V. The references should be consulted for general information in constructing and using a numerical integration package for ordinary differential equations.

II. INTEGRATION SCHEME IN SPECIAL RELATIVITY

This section outlines how equations of motion can be numerically integrated in special relativity. Numerical integration of relativistic motion parallels the Newtonian method closely. Time is the independent integration parameter integrating functions in a static inertial reference frame. The reference frame usually expresses vectors in a multi-dimensional Cartesian coordinate system. The task within special relativity is to numerically integrate coupled vectorial first-order differential equations of motion that incorporate the forces applied to the moving body under analysis. These differential equations use definitions contained in special relativity, and reduce to the corresponding Newtonian equations for small velocities relative to the speed of light. The end results of the integration are typically kinematic vectors such as location and velocity of the body given as a function of time and expressed within the integration reference frame.

In dynamics analyses, acceleration is the time derivative of velocity which, in turn, is the time derivative of location. For integration of Newtonian body motion, location and velocity vectors usually make up the dependent integration functions g_i of

Eq. (1); these functions are integrated through derivative functions f_i designated as velocity and acceleration, respectively. Newtonian integration methods normally involve summing force vectors $\boldsymbol{F}_i$ on a body at a particular instant in time. This summation is then used to estimate the acceleration vector $\boldsymbol{a}$ of the body at that time with the equation

$$\sum F_i = \mathrm{ma} \tag{2}$$

The variable m in the above equation is the instantaneous mass of the body undergoing the force analysis. Applied forces are related to the derivative of velocity (i.e., acceleration) through this equation, and the dependent integration functions are integrated.

In special relativity, Eq. (2) is no longer valid due to the relativistic definition of force given by Eq. (25). A replacement to Eq. (2) can be developed from this force definition,

$$\frac{dp}{\mathrm{dt}} = \sum F_i \tag{3}$$

The vector $\boldsymbol{p}$ in the above equation is the body momentum defined by Eq. (17). The variable t is the integration parameter time. This equation assumes that relativistic force definitions allow forces to be combined in a vectorial fashion. Applying Eq. (3), momentum becomes a dependent integration function g_i of Eq. (1) in place of the velocity vector used in Newtonian mechanics. Momentum is not a kinematic quantity, and what is primarily required for definition of body motion are kinematic vectors such as location and velocity. However, momentum incorporates the velocity vector in its definition (Eq. (17)). This definition can be inverted to express velocity as a function of momentum,

$$\frac{dx}{dt} = \frac{p}{(m^2 + p^2/c^2)^{1/2}} \tag{4}$$

$$g_p = (1 + p^2/m^2c^2)^{1/2} \tag{5}$$

This *gamma* term occurs frequently in special relativity and is defined in Eq. (10).

A. Similarities Between Newtonian and Special Relativity Integrations

Two important characteristics should be noted in the numerical integration of Newtonian motion. These characteristics are equally applicable in special relativity. (1) The variable time is the primary integration variable used in advancing the equations of motion. (2) The integration is performed in one inertial reference frame.

In special relativity, changes in time are no longer constant through transformation of reference frames. However, in a static integration reference frame, time is still the independent parameter of integration for special relativity just as in Newtonian mechanics. The parameter time applies universally to all simultaneous events throughout the inertial reference frame. The equations of motion use location and momentum as integrals of applied force with the single integration parameter time. Velocity and force are defined as time derivatives of location and momentum, respectively. Provided that the force functions can be described in terms of location and its time derivatives at a single instant in time, the equations of motion are still ordinary differential equations. These equations can be integrated equivalently in special relativity with integration methods such as Runge-Kutta just as they are used in Newtonian mechanics.

In Newtonian integration of motion, the integration is performed in one inertial reference frame. When the process is transferred to special relativity, none of its complicated transformation equations are used in direct integration of the equations of motion. Transformation to other reference frames may be performed, say to a body in motion, but usually to define derivatives such as body forces. Once defined, the derivatives are always transformed back into the integration reference frame for integration. In Newtonian physics, transferring quantities between inertial reference frames, such as time, force, relative distance, or relative velocity, is an elementary algebraic process at most. These transformations no longer hold in special relativity, and anytime conditions are transferred to or from a body in motion, more complicated transformations are employed. Vector forms of these transformations are given in Sec. III for several quantities.

B. Special Considerations

It may be implied that special relativity cannot accurately predict accelerated motions subject to forces, since reference frames are only defined relative to inertial frames with constant velocity. However, within and between reference frames, vectors and their transformations are defined for displacement and all its time derivatives. All of these vectors can be meaningfully discussed in the context of special relativity and the Lorentz transformations[2]. Force is defined as the time derivative of momentum; the existence of this definition implies that special relativity encompasses accelerated motion. Within the integration scheme discussed above, velocity can be expressed in terms of momentum and rest mass through the definition of momentum. Velocity is defined as the time derivative of location, and both force and velocity can be integrated within an inertial frame with the independent parameter time of the inertial frame. In modeling accelerating motion, the integration scheme of this section is based on definitions contained in special relativity.

The rest frame of a body undergoing acceleration due to force is not an inertial reference frame. Therefore, it may be implied that any quantities relating to or derived from the rest frame of an accelerating body cannot be accurately expressed. However, a stratagem can be used whereby the proper time frame of the body can be thought of as passing through infinitely many successive inertial frames[3]. These inertial frames are at constant velocity as the body motion changes with respect to some overall constant reference frame. In analyzing and transforming quantities from the rest frame of a body accelerating, velocity of the body is taken as constant, and transformation equations can then be applied. Even though a body is under acceleration, it is subjected to forces and accelerations which can be evaluated within its rest frame.

III. VECTOR TRANSFORMATIONS

This section derives vector equations which transform quantities from an initial reference frame to a frame moving with a constant specified velocity. Transformations of vectors in special relativity, such as position and momentum, are traditionally defined in text by equations of Cartesian coordinates with a preferred orientation, and the x-axis is usually oriented toward the velocity vector of the moving reference frame. Matrices of the four-vector space-time are also given this preferred orientation leaving the three-dimensional space rotations as an additional process. In simulating special relativity dynamics in computer analyses, the modeling is usually multi-dimensional; body movement is rarely anticipated to be aligned with the chosen coordinates. Fortunately, both three- and four-vector transformations can be expressed in vector equations independent of axis orientation. Four-vector transformations lose their coherence in these expressions, but numerical analyses are streamlined because space rotations are eliminated.

The vector transforms can be proved algebraically equivalent to the matrix transform process. A vector transform is essentially a matrix space rotation of the vector to an axis aligned with velocity, then, a transformation boost into moving coordinates, and an anti-rotation back to the original orientation. The vector transforms maintain the three-dimensional space orientation of the vector. Except for a special case with the transformation velocity equal to zero, the vector transforms are just as applicable as the preferred-axis or matrix transforms. As they should, the vector transforms reduce to the preferred-axis transforms when the transformation velocity is given only a single coordinate component in the expressions.

Vector transforms for several quantities are scattered through the literature. The vector transforms of location and time are given in a reference[3], and other vector transforms can be derived directly from these equations. The vector transforms for location and velocity have been given in another reference[4]. However, it is sometimes easier to derive the transforms in the preferred-axis orientation, then make vector substitutions to get the vector expressions. Preferred-axis transforms have been derived for time, location, energy, momentum, velocity, acceleration, and force[2]. Vector equivalents are substituted into these expressions to derive the vector transforms.

In the following preferred-axis transforms, the x-axis is always aligned with the velocity vector $\boldsymbol{v}$ of the moving reference frame. With a set of preferred-axis transformation equations, the following vector substitutions are used to convert the equations into vector form. Given any vector $\boldsymbol{w}$ defined as

$$v = v_x \hat{\imath}$$
$$w = w_x \hat{\imath} + w_y \hat{\jmath} + w_z \hat{k} \tag{6}$$

then

$$(w * v) = w_x v_x \tag{7}$$

$$\frac{(w*v)}{v^2} v = w_x \hat{\imath} \tag{8}$$

$$w - \frac{(w*v)}{v^2} v = w_y \hat{\jmath} + w_z \hat{k} \tag{9}$$

The velocity vector $\boldsymbol{v}$ remains constant in the above substitutions and is the transformation velocity. The vector $\boldsymbol{w}$ changes to location $\boldsymbol{r}$, momentum $\boldsymbol{p}$, or other vectors as required in the transformation equation terms. On the left side of the equations are vector expressions which are equivalent to the preferred-axis terms on the right. Eq. (7) simply states that the vector dot product of $\boldsymbol{w}$ and $\boldsymbol{v}$ is the projection of $\boldsymbol{w}$ onto the x-axis which is aligned with $\boldsymbol{v}$. Eq. (8) takes this projection, gives it the direction vector of $\boldsymbol{v}$, then divides by v^2 canceling the velocity magnitudes. Eq. (9) subtracts Eq. (8) from Eq. (6) giving the part of $\boldsymbol{w}$ perpendicular to the velocity

v. Of course, from the above equations, velocity cannot be directly evaluated at zero, but no transformations are needed in this case.

In the vector transforms, the following terms are given special notations since they appear numerous times.

$$\gamma = 1/\sqrt{1 - v^2/c^2} \tag{10}$$

$$\delta = \left(1 - (u * v)/c^2\right) \tag{11}$$

$$\sigma = \frac{(1 - 1/\gamma)c^2}{v^2} \tag{12}$$

All of these terms incorporate ratios and are hence unitless. The variable *c* is the speed of light in a vacuum. In Eq. (11), the vector ***u*** is the in-frame velocity of the body on which the vector acts in unprimed coordinates, and may not necessarily be equivalent to the transformation velocity ***v***. By convention, the primed quantities in the following equations are the transformed vectors and scalars expressed in the transformation reference frame. The unprimed quantities are expressed in the initial reference frame. In the forms that the transformation equations are given, more efficient coding can often be achieved by factoring and/or canceling *gamma* and c^2 terms.

A. Transformation of Time and Location

The preferred-axis transformation equation of time is[2]

$$t' = \gamma(t - r_x v/c^2) \tag{13}$$

In the above equation, the variables t and t' are initial and transformed times, respectively. The variable r_x is the projection of the location vector ***r*** onto the transformation velocity ***v*** direction. Substituting Eq. (7) for the term $r_x v$, the vector form of the above equation becomes

$$t' = \gamma\left(t - (\mathrm{r} * v)/c^2\right) \tag{14}$$

Eq. (13) has now been put into vector form with no preferred orientation of axes.

The preferred-axis transformation equations of location are[2]

$$r'_x = \gamma(r_x - vt) \qquad r'_{y,z} = r_{y,z} \tag{15}$$

Eq. (13) and Eq. (15) are defined such that the origins of the two reference frames coincide when time t equals zero. Using Eq. (8) and Eq. (9) for the location vector ***r***, the vector form of the above equations is

$$r' = r + \gamma v\left[\frac{(\mathrm{r} * v)}{c^2}\sigma - \mathrm{t}\right] \tag{16}$$

After vector substitutions are made, the equations in Eq. (15) are simply added together. In the above equations, the term vt has been directly turned into a vector. The term σ cancels the x-

coordinate term from the vector ***r*** addition and adds a *gamma*-r_x term in its place.

B. Transformation of Momentum and Energy

The length of the four-vector momentum-energy (***p***,*E/c*) is conserved in transformation just as the length of the four-vector space-time (***r***,*ct*). Except for factors of *c*, momentum and energy are transformed per Eq. (15) and Eq. (13), respectively. Total energy *E* and momentum ***p*** of a body moving with velocity ***u*** are defined by the equations[2]

$$E \equiv \gamma_u \, mc^2 \qquad \mathrm{p} \equiv \gamma_u \, \mathrm{mu} \tag{17}$$

where the variable *m* is the rest mass of the body. In the above equations, the term $gamma_u$ is a function of the in-frame body velocity ***u***. The vector transforms for energy and momentum are

$$E' = \gamma[E - (p * v)] \tag{18}$$

$$p' = p + \frac{\gamma v}{c^2}[(p * v)\sigma - E] \tag{19}$$

C. Transformation of Velocity

The preferred-axis transformation equations for velocity are[2]

$$u_{x'} = \frac{u_x - v}{(1 - u_x v/c^2)} \qquad u_{y',z} = \frac{u_{y,z}}{\gamma(1 - u_x v/c^2)} \tag{20}$$

where the vector ***u*** is the velocity to be transformed. Using Eq. (7) through Eq. (9) for the vector ***u***, the vector transformation equation becomes

$$u' = \frac{1}{\delta}\left[\frac{u}{\gamma} + \mathrm{v}\left[\frac{(\mathrm{u} * v)}{c^2}\sigma - 1\right]\right] \tag{21}$$

In the above equation, Eq. (11) has been substituted for the denominators of Eq. (20).

D. Transformation of Acceleration

The preferred-axis transformation equations for acceleration are[2]

$$a_{y',z} = \frac{a_{y,z}}{\gamma^2 (1 - u_x v/c^2)^2} + \frac{u_X, a_x}{c^2\gamma^2 (1 - u_x v/c^2)^3} \tag{22}$$

where vector ***a*** is the acceleration of the body and vector ***u*** is the velocity of the body. Using Eq. (7) through Eq. (9) for vector ***a*** and Eq. (7) and Eq. (9) for vector ***u***, the vector transform is

$$a' = \frac{1}{\delta^2\gamma^2}\left[\mathrm{a} - \frac{(\mathrm{a*v})}{\delta c^2}[v\sigma - \mathrm{u}]\right] \tag{23}$$

E. Transformation of Force

The force vector ***F*** in special relativity is defined by the equation[2]

$$\mathrm{F} \equiv \frac{\mathrm{dp}}{\mathrm{dt}} \tag{24}$$

where the momentum vector ***p*** is defined by Eq. (17). This definition is not constrained to the proper time reference frame. The preferred-axis transformation equations for the force vector are[2]

$$F_{x'} = \frac{F_x - (v/c^2)(\mathrm{F} * \mathrm{u})}{(1 - u_x \mathrm{v}/c^2)} \qquad F_{y',z} = \frac{F_{y,z}}{\gamma(1 - u_x \mathrm{v}/c^2)} \tag{25}$$

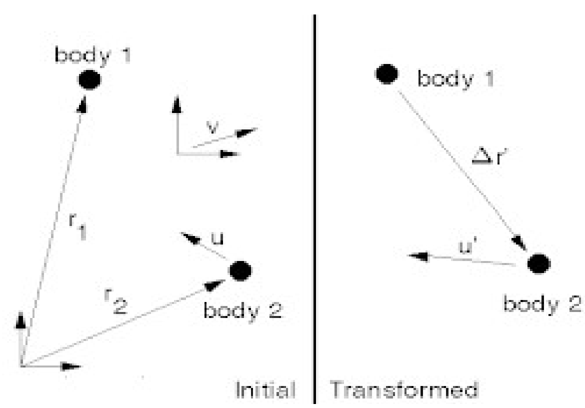
where the vector ***u*** is the velocity of the body undergoing the force ***F***. Using Eq. (8) and Eq. (9) for vector ***F*** and Eq. (7) for vector ***u***, the vector transform is

$$F' = \frac{1}{\delta}\left[\frac{F}{\gamma} + \frac{v}{c^2}[(\mathrm{F} * v)\sigma - (\mathrm{F} * u)]\right] \tag{27}$$

F. Location Difference Transformation

In the modeling of multiple body dynamics in special relativity, a problem often encountered is determining distances between bodies moving at different velocities. In many cases, the forces between bodies are dependent on these distances. If the need arises to transform the distance difference between reference frames, an equation must be developed to calculate the transformation. A requirement in determining the transformation is that simultaneity be applicable for the location difference vector before and after the transformation.

The transformation of distance between two bodies is not a simple problem because simultaneity of two events in different locations may not be valid in moving reference frames. In examining time and distance transformation equations (Eq. (14) and Eq. (16)), time changes as distances change. The configuration of differently moving bodies in one reference frame is not the same in a frame moving with respect to it even though a single point simultaneity of reference occurs between the two frames. A reference frame moving with respect to a specific configuration sees the bodies at different points along their trajectories and hence a different configuration.



Consider two bodies, body 1 at position $\boldsymbol{r}_1$ moving with some velocity, and body 2 at position $\boldsymbol{r}_2$ moving with velocity $\boldsymbol{u}$ as shown in Fig. 1. The extent of this derivation is to find the location vector of body 2 relative to body 1 in a reference frame moving with velocity $\boldsymbol{v}$. An additional constraint is that the position of body 1 be simultaneous in both the initial and transformed reference frames. Because body 1 has been chosen to be simultaneous in both reference frames, its velocity is immaterial. Body 2 is assumed to have a constant velocity trajectory for this derivation. This body 2 velocity will take part in assuring a simultaneous location difference in the transformed reference frame.

The time transformation of location vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ is given by Eq. (14); the location transformation is given by Eq. (16). In examining the equations, the transformed locations $\boldsymbol{r}_1'$ and $\boldsymbol{r}_2'$ do not necessarily occur at the same time. If body 1 is at $\boldsymbol{r}_1$ and body 2 is at $\boldsymbol{r}_2$ in some reference frame at the same time *t*, and both are moving along different trajectories, they are not necessarily in the same locations along their trajectories in a transformed reference frame simultaneously, and vice versa. For the transformations to occur simultaneously in the primed coordinates, times should be assumed different in unprimed coordinates. Body 2 can be displaced along its linear trajectory in unprimed coordinates an amount of time sufficient to allow for simultaneity in the primed coordinates.

Let the variable *delta t* represent the amount of time difference between two events, body 1 at $\boldsymbol{r}_1$ and body 2 at some other location $\boldsymbol{r}_{2a}$ along its trajectory. Because the body 2 trajectory is assumed linear, the location vector $\boldsymbol{r}_{2a}$ is given as a function of its velocity $\boldsymbol{u}$ and *delta t* by the equation

$$r_{2a} = r_2 + u\Delta t \quad @ \quad t + \Delta t \tag{28}$$

The variable *t* is some time quantity which applies to simultaneous events body 1 at $\boldsymbol{r}_1$ and body 2 at $\boldsymbol{r}_2$ in the initial reference frame of the analysis. The event body 2 at location $\boldsymbol{r}_{2a}$ occurs at time *t* + *delta t* in the initial reference frame. With a new vector location $\boldsymbol{r}_{2a}$ defined for body 2, Eq. (16) transforms the location vectors of the two bodies as

$$r'_1 = r_1 + \gamma v \left(\frac{(r_1 * v)}{c^2}\sigma - t\right) \tag{29}$$

$$r'_2 = r_2 + u\Delta t + \gamma v \left(\frac{(r_2 * v) + (u * v)\Delta t}{c^2}\sigma - (t + \Delta t)\right) \tag{30}$$

The variables *gamma* and σ in the above equations are functions of the transformation velocity $\boldsymbol{v}$ as defined by Eq. (10) and Eq. (12), respectively. The vector $\boldsymbol{r}_1'$ is the transformed location $\boldsymbol{r}_1$ of body 1. The vector $\boldsymbol{r}_2'$ is the transformed location of vector $\boldsymbol{r}_{2a}$ with body 2 in its displaced position. In Eq. (30), Eq. (28) has been transformed through substitution into Eq. (16). Subtracting Eq. (29) from Eq. (30) gives the vector location difference between bodies 1 and 2 relative to a reference frame moving with velocity $\boldsymbol{v}$.

$$\Delta r' = u \left[\frac{(\Delta \mathrm{r} * v) + (\mathrm{u} * v)\Delta t}{c^2}\sigma - \Delta \mathrm{t}\right] \tag{31}$$

In the above equation, the term *delta* $\boldsymbol{r}$ is location vector $\boldsymbol{r}_1$ subtracted from vector $\boldsymbol{r}_2$ in the unprimed coordinates. At this point, *delta t* is unknown; however, it can be solved for through the use of the time transformation equation (Eq. (14)). Transforming the times of events body 1 at $\boldsymbol{r}_1$ and body 2 at $\boldsymbol{r}_{2a}$, the equations can be stated as

$$t'_1 = \gamma[t - (r_1 * v)/c^2] \tag{32}$$

$$t'_1 = \gamma\left[(t + \Delta t) - \frac{[(r_2 * v) + (u * v)\Delta t]}{c_2}\right] \tag{33}$$

The value *delta t* is chosen such that both events transform to the same time t_1' in the primed coordinates. Though it should be noted that the transformed time is for body 1 at $\boldsymbol{r}_1$, not for body 2 at $\boldsymbol{r}_2$. Body 2 is transformed at a location $\boldsymbol{r}_{2a}$ and at a time t + *delta t* somewhere along its trajectory. Setting Eq. (32) and Eq. (33) equal (thereby eliminating t_1'), *delta t* can be solved for

$$\Delta t = \frac{(\Delta \mathrm{r} * v)}{(1 - (\mathrm{u} * \mathrm{v})/c^2)c^2} = \frac{(\Delta \mathrm{r} * v)}{\delta c^2} \tag{34}$$

The term *delta t* of the above equation represents the amount of time difference needed between the two events, body 1 at $\boldsymbol{r}_1$ and body 2 somewhere along its trajectory from $\boldsymbol{r}_2$. This difference allows for the bodies to be simultaneous in a reference frame moving with velocity $\boldsymbol{v}$. The term *delta* $\boldsymbol{r}$ in the above equation is location vector $\boldsymbol{r}_2$ minus vector $\boldsymbol{r}_1$ in the unprimed coordinates.

Eq. (31) is solved for *delta* ***r'*** in terms of the given locations and velocities by substituting Eq. (34) directly into Eq. (31). After some algebra is performed with several cancellations, the result simplifies to

$$\Delta r' = \Delta r - \frac{(\Delta r * v)}{\delta c^2}[v\sigma - u] \tag{35}$$

Given a vector location difference *delta* ***r*** between two bodies (one moving with velocity ***u***) in an initial reference frame, the above equation transforms the location difference to a reference frame moving with velocity ***v***. The body not moving with velocity ***u*** is simultaneous in both reference frames, and its velocity does not affect the transformation. The important assumption here is that the body velocity ***u*** is constant. In applying the above equation in numerical analyses, errors will be small in relatively low velocities and small distances, or low accelerations with high velocities.

G. The σ Term at Zero Velocity

The σ term is defined in Eq. (12). Even though the term has velocity in the denominator, the expression is well behaved at velocity equal to zero. Expanding the reciprocal *gamma* term from Eq. (10) in a Taylor series about *v* equal to zero leads to

$$(1 - v^2/c^2)^{1/2} = 1 - \frac{v^2}{2c^2} + \ldots \tag{36}$$

Subtracting this expression from unity and multiplying by c^2/v^2 gives the term evaluated at zero.

$$\sigma \,|_{v=0} = \frac{1}{2} \tag{37}$$

When the transformation equations are coded and the σ term is used, a conditional will have to be included for velocity evaluations equal to zero.

IV. DEFINING FORCES FOR INTEGRATION

This section discusses forces in special relativity and defines equations applicable to forces for use in numerical integration of body motion. Because force has a definition in special relativity (Eq. (25)) algebraically different from Newtonian mechanics, many forces have to be redefined, and some lose their validity. Care must be taken in expressing forces and incorporating them into the equations of motion. Force is not related simply to kinematic vectors such as acceleration and velocity as in Newtonian mechanics (Eq. (2)). Force and acceleration vectors do not transform equivalently between inertial reference frames, and applied force is not always parallel to the resultant acceleration. Furthermore, force definitions must take into account propagation effects from the source to the body experiencing the force. Newton's third law of action-reaction incorporating "action at a distance" no longer applies in general[2]. Finally, this article does not consider rotations and torques. All forces are assumed to act along a line through the center of mass of the body.

The coulomb force exerted on a point charge by another point charge is given in Sec. IV A. The force expression is derived for application in the integration reference frame. Because the coulomb force equation is derived from the reference frame in which the charged body exerting the force is stationary, no magnetic forces due to moving charges need be considered. Magnetic forces are taken into account by the force transformation equation (Eq. (27)). In the development of the coulomb force equation, the charge exerting the force is assumed to have a linear trajectory.

Sec. IV B gives an equation for apparent force on a body due to acceleration from ejected mass. This expression is only valid in the integration reference frame. If the condition of variable mass applies, mass of the body becomes a dependent integration function g_i of Eq. (1) in the numerical integration, and must be supplied a mass derivative function f_i which is discussed in Sec.

IV D. A one-dimensional case of a variable mass system is solved analytically in Sec. V B, and results are compared with numerical computations using methods and equations outlined in this article.

Forces may depend on the passage of time with respect to a moving body. Unlike Newtonian mechanics, the rate of time flow changes in a moving body, and is different from the time increments being integrated in the integration reference frame. To determine the reading of a clock moving with the body, a separate integration variable is used which is different from the independent integration parameter t. In applying Eq. (1) of the numerical integration, the moving clock time is expressed as a dependent integration function g_i; the time derivative, given in Sec. IV C, becomes its derivative function f_i.

A. Coulomb Force

Special relativity is based on a postulate involving electromagnetism, and the only field force defined in special relativity is the electromagnetic force. For completeness of treatment, the Coulomb force is developed here such that it can be evaluated for a moving charged particle with other charged particles moving around it. The Coulomb force can then be applied with this formulation within the integration reference frame in computer analyses.

The Coulomb force of electrostatics is traditionally defined as an inverse square law force exerted by a stationary source point charge on another stationary point charge at a given distance from it. For this discussion, the charge creating the electric field is labeled the source charge; the charge subject to the force of the field is labeled the test charge. The extent of this derivation is to find the Coulomb force produced on the test charge by the source charge. The Coulomb force can be further extended to encompass a test charge moving with a given velocity in relation to a stationary source charge[2]. In the reference frame of the source charge, the Coulomb force is then expressed by the equation

$$F_{c'} = \frac{kq_s q_t}{\Delta r'^3} \Delta' r \qquad (38)$$

In the above equation, the vector $\boldsymbol{F}_c'$ is the Coulomb force exerted by the source charge on the test charge at the point of the test charge; the force vector is given in the reference frame of the source charge. The variables q_s and q_t are the charge quantities for the source and test charges, respectively. The vector *delta* $\boldsymbol{r}'$ is the distance from the source charge to the test charge in the source charge reference frame. The variable *k* is the Coulomb force proportionality constant which in meter-kilogram-second (MKS) units is $1/4\pi\varepsilon_0$, ε_0 being the permittivity constant. Based on the extension of the Coulomb force, the above equation holds regardless of the speed of the test charge in relation to the source charge.

As stated in Sec. II, forces must be expressed in the integration reference frame for numerical integration. An integration value using Eq. (38) can be calculated by first transforming integration reference frame velocities and distances of the charges into the source charge reference frame with vector equations of Sec. III. In using the location difference transformation equation (Eq. (35)), a constant velocity constraint is placed on the source charge trajectory. The Coulomb force is calculated through Eq. (38) with the transformed location difference vector. Then, the force is transformed back into the integration reference frame with the force transformation equation (Eq. (27)) using the negative of the source charge velocity and the transformed test charge velocity. To simplify calculations, these substitutions can be made algebraically to directly express the Coulomb force in a single equation in terms of integration reference frame parameters.

The force transformation equation (Eq. (27)) is used with force vector ***F*** (the vector to be transformed) given by Eq. (38), and the transformation velocity is the negative of the source charge velocity ***v*** expressed in the integration reference frame. The test charge velocity ***u*** is transformed to the source charge rest frame with Eq. (21), and this equation is substituted directly into Eq. (27). Making these substitutions, the force transformation is

$$F'_c = \frac{\delta}{[(u-v)^2-u^2]/c_2}\left[\frac{F_c}{\gamma\delta} + v\frac{2(F_c*v)\sigma}{c^2}\right] \qquad (39)$$

The above equation transforms the force ***F'*** on a body moving with velocity ***u*** from a reference frame moving with velocity ***v***, the source charge velocity. The force vector ***F'*** is expressed in the transformation frame moving with the source charge, and is given by Eq. (38). The vectors ***F***, ***u***, and ***v*** are expressed in the integration frame. The *gamma* and σ terms are functions of the source charge velocity as defined in Eq. (10) and Eq. (12), respectively; the δ term is a function of both source and test charge velocities as defined in Eq. (11).

The distance between source and test charges must be expressed in the source charge rest frame to apply Eq. (38). To take into account propagation effects, the source charge trajectory must be known, and is assumed linear for this derivation. Then, Eq. (35) can be used to transform the location difference between the two charges. In applying this equation, the Coulomb force is evaluated at the instantaneous time and location of the test charge in the integration reference frame. The source charge location in the integration reference frame is displaced along its trajectory to allow for location difference simultaneity in the source charge rest frame. In using Eq. (35), the transformation velocity is the same as the velocity along which the source charge is displaced, i.e., the source charge velocity. The location difference transformation, for this case, sets the in-frame velocity ***u*** equal to the transformation velocity ***v***. The transformation is

$$\Delta r' = \Delta r + v\,\frac{\gamma(\Delta r * v)}{c^2} \qquad (40)$$

In the above equation, for application with Eq. (38), the term *delta* ***r*** is the test charge location minus the source charge location. The terms *gamma* and σ are functions of the source charge velocity ***v*** which is the same as the transformation velocity. Eq. (40) transforms the location difference between source and test charges to the rest frame of the source charge such that the location of the test charge is simultaneous in both reference frames.

Carrying through the substitutions of Eq. (40) into Eq. (38) and this expression into the force transformation equation (Eq. (39)), the Coulomb force can be expressed in the integration reference frame as

$$F_c = \frac{k\gamma q_s q_t}{R^3}\left[\Delta \mathrm{r}\left(1 - \frac{(\mathrm{u} * v)}{c^2}\right) + \mathrm{v}\,\frac{(\Delta \mathrm{r} * u)}{c^2}\right] \qquad (41)$$

with

$$R = \left(\Delta r^2 + \gamma^2 (\Delta \mathrm{r} * v)^2 / c^2\right)^{1/2} \qquad (42)$$

The above equations express the Coulomb force $\boldsymbol{F}_c$ on the test charge given the distance between the two charges *delta* ***r***, the velocity ***v*** of the source charge, the velocity ***u*** of the test charge, and the charge quantities q_s and q_t in the integration reference frame. The *gamma* term is a function of the source charge velocity which is given in Eq. (10). As it should, Eq. (41) reduces to Eq. (38) when the velocity of the source charge approaches zero.

The above equations are only valid for source charges moving with uniform velocity with respect to an inertial reference frame. However, the equation can be used as an approximation for greatly accelerating charges in numerical calculations if distances are relatively short or velocities are relatively small. The above equations can be coded more efficiently in an alternate form if the *gamma*^{2} term is factored out of the parentheses of the *R* term in Eq. (42).

Eq. (41) encompasses the magnetic effects of the moving source charge. The equation can be put into the cross product form[2] (in MKS units)

$$F_c = q_t(\mathrm{E} + \mathrm{u} \times \mathrm{B}) \qquad \mathrm{B} = \frac{1}{c^2}(\mathrm{v} \times \mathrm{E}) \tag{43}$$

In the above equations, the vector ***B*** is the magnetic field of the source charge. The vector ***E*** is the electric field of the source charge given by the equation

$$\mathrm{E} = \frac{k\gamma q_s}{R^3}\Delta r \tag{44}$$

It may appear that Eq. (41) is not based on a retarded potential and does not take into account field propagation times of the moving source charge. However, the equation was derived from the rest frame of the source charge in which no retarded potentials are needed. In this inertial rest frame, the electric field is spherically symmetrical about the source charge and is static.

Any force transformations from this rest frame make the use of a retarded potentials unnecessary.

B. Acceleration on Systems of Variable Mass

This section develops an equation describing the acceleration exerted on a variable mass system due to the ejection of mass from the system at a given loss rate and with a given vector velocity. The equation solves the rocket problem casted in special relativity. The final equation gives a relation between acceleration, mass ejection velocity, and mass ejection rate. For the rocket problem, the velocity derivative as a function of mass has been solved in one dimension as[3]

$$m \frac{dv}{dm} + \frac{w}{\gamma^2} = 0 \tag{45}$$

The above equation describes the velocity of a variable mass system such that no external forces act on the system. The variable *m* is the instantaneous rest mass of the rocket, and *w* is the velocity of the ejected mass relative to the rocket. The variable *v* is the velocity of the rocket with respect to an inertial reference frame, and *gamma* is a function of this velocity defined by Eq. (10).

Eq. (45) can be developed in a three-dimensional vector form by considering the four-vector momentum-energy before and after mass is ejected in a time period *delta t*. This four-vector is conserved. The energy conservation equation is substituted into the momentum vector conservation equation, and the limit is taken as *delta t* goes to zero. Mass ejected is related to mass lost by the rocket through the energy needed to accelerate the ejected mass in the rest frame of the rocket. The ejected mass momentum is related to its momentum relative to the rocket through the momentum transformation equation (Eq. (19)). However, this section will simply use Eq. (45) to develop an expression for rocket acceleration in the rest frame of the rocket. The acceleration will then be transformed into a three-dimensional vector equation in the integration reference frame using the acceleration transformation equation (Eq. (23)).

Applying the chain rule with mass and velocity both a function of time, Eq. (45) can be put into the form

$$\gamma^2 \, m \, \frac{dv}{dt} = - \, w \, \frac{dm}{dt} \tag{46}$$

When the above equation is used in the rest frame of the rocket, velocity *v* equals zero and the *gamma* term goes to unity. Furthermore, using vector notation, ejected mass velocity has the opposite direction (and hence a negative value) from the change in rocket velocity. Then, in vector form for the rocket's proper time reference frame, Eq. (46) can be written as

$$a' = \frac{dv\prime}{d\tau} = \frac{-dm}{d\tau}\frac{w}{m} \tag{47}$$

The above equation gives the acceleration vector ***a'*** due to ejected mass in proper time τ of the moving body; all quantities are expressed in the rest frame of the rocket. The Newtonian solution of the rocket problem[5] agrees with this solution. Eq. (47) is transformed to the integration reference frame using the acceleration transformation equation (Eq. (23)) where the transformation velocity is equal to the negative of the rocket velocity ***v***. In Eq. (23), the in-frame velocity ***u*** of the rocket is zero, and the δ term goes to unity. The acceleration vector transformation equation from proper time is

$$a = \frac{1}{\gamma^2}\left[a' - v\sigma\frac{(a\prime * v)}{c^2}\right] \tag{48}$$

The above equation expresses the integration reference frame acceleration ***a*** of the rocket moving with velocity ***v*** in terms of its rest frame acceleration ***a'***. The *gamma* and σ terms in the above equation are functions of the rocket velocity defined by Eq. (10) and Eq. (12), respectively. Substituting Eq. (47) into Eq. (48) leads to

$$a = \frac{1}{\gamma^2 m}\frac{dm}{d\tau}\left[v\sigma\frac{(w * v)}{c^2} - w\right] \tag{49}$$

The above equation gives an expression in the integration reference frame for acceleration ***a*** of a rocket traveling with velocity ***v*** and ejecting mass with velocity ***w*** relative to the rocket. The mass loss rate is expressed in the rest frame of the rocket.

As outlined in Sec. II, momentum is a dependent integration function with force supplied as its derivative function. To model variable mass systems in this framework, the acceleration in Eq. (49) has to be expressed as an apparent force. The definition of force is given in Eq. (25). Carrying through the differentiation of this equation, the force vector ***F*** as a function of acceleration ***a*** is given as

$$\mathrm{F} = \gamma\left[\ \mathrm{ma} + \mathrm{v}\left[\frac{\mathrm{dm}}{\mathrm{dt}} + \frac{\gamma^2 m(\mathrm{a} * \mathrm{v})}{c^2}\right]\right] \qquad (50)$$

The acceleration of the rocket is given in Eq. (49). The mass derivative of Eq. (50) must also be converted to a proper time derivative using Eq. (54). Substituting these expressions into the above equation, and after some algebra, the apparent force on a variable mass system becomes

$$\frac{dp}{\mathrm{dt}} = \frac{\mathrm{dm}}{d\tau}\left[\frac{w}{\gamma} + \mathrm{v}\left[\frac{(\mathrm{w} * \mathrm{v})\sigma}{c^2} + 1\right]\right] \qquad (51)$$

The above equation gives the time rate of change of momentum of a variable mass system due to acceleration from ejected mass. Both Eq. (49) and Eq. (51) reduce to their Newtonian counterparts[5] when rocket velocity $\boldsymbol{v}$ is small relative to the speed of light. Using the integration scheme discussed in Sec. II, Eq. (51) can be integrated to determine the dynamics of a variable mass system. However, the equation does not define a force or a vector which transforms like a force into another reference frame. The equation calculates a change in momentum due to an acceleration.

C. Proper Time Derivative

Numerical analyses integrate variables in a constant reference frame other than the proper time frame of the moving body. To determine the passage of time for the moving body, the derivative of proper time with respect to the integration reference frame must be evaluated. With the proper time derivative supplied and integrated by computer, the reading of a clock can be calculated as it is carried with the moving body.

Differentiating Eq. (14) for time transformation and holding the transformation velocity $\boldsymbol{v}$ constant, the proper time derivative is

$$\frac{d\tau}{dt} = \gamma(1 - (u * v)/c^2)|_{u=v} = \frac{1}{\gamma} \tag{52}$$

The above equation is evaluated at the in-frame body velocity ***u*** equal to the transformation velocity ***v*** which defines the proper time reference frame. When the reciprocal *gamma* term is evaluated at the velocity of the moving body and integrated, passage of time can be calculated on the moving body. As the equation shows, the change in time slows down for moving bodies relative to an inertial reference frame.

D. Variable Mass Derivative

In calculating the dynamics of a variable mass system with computer methods, body mass changes, and the mass variable must be supplied a derivative for its integration. The mass variable is integrated in references frames other than proper time; therefore, a transformation of the mass derivative from proper time may be necessary.

The mass derivative is transformed from proper time τ to some other reference frame time t in the usual chain rule fashion

$$\frac{dm}{dt} = \frac{dm}{d\tau}\frac{d\tau}{dt} \tag{53}$$

where the variable m is the time dependent rest mass of the body. The derivative of proper time with respect to some other time is given by Eq. (52). Substituting this expression into Eq. (53), the transformation of the mass derivative from proper time becomes

$$\frac{dm}{dt} = \frac{1}{\gamma}\frac{dm}{d\tau} \tag{54}$$

As the above equation indicates, mass rates of change slow down in reference frames moving with respect to the proper time reference frame.

V. COMPARISON OF NUMERICAL AND ANALYTICAL SOLUTIONS

Equations describing body dynamics in special relativity are lengthy and complicated. However, specific cases can be solved to allow data comparison between calculations of computer methods and analytical solutions. In this section, three specific cases of analytical solutions of body motion in special relativity are derived. Data plots compare these solutions with numerical predictions that apply methods and equations discussed in this article.

The integration algorithm used in the following numerical calculations was a Runge-Kutta-Fehlberg Method with modifications by Cash-Karp[1]. The algorithm employed an adaptive step-size control sixth-order evaluation with embedded fifth-order evaluation for error checking.

A. Body Subjected to Constant Force

The one-dimensional equations of motion have been solved for a body of constant rest mass subjected to a constant force in special relativity[3]. From the definition of force (Eq. (24)) as the time derivative of momentum, the following equation can be written concerning the body motion,

$$\frac{d}{dt}\gamma v = \frac{F}{m} = f \tag{55}$$

In the above equation, the variable *v* is the velocity of the body in some inertial reference frame; the term *gamma* is a function of this velocity given by Eq. (10). The variable *m* is the rest mass of the body undergoing a constant force *F*. The term *f* is the force per unit rest mass in the analysis. Integrating Eq. (55) with respect to time *t*, then solving for velocity *v* leads to

$$v = \frac{c(ft + \alpha)}{(c^2 + (ft + \alpha)^2)^{1/2}} \tag{56}$$

In the above equation, the variable α is a constant of integration. The variable *c* is the speed of light. Integrating Eq. (56) for position of the body, the answer becomes

$$x - x_c = \frac{c}{f}\left[(c^2 + (ft + \alpha)^2)^{1/2} - (c^2 + \alpha^2)^{1/2}\right] \qquad (57)$$

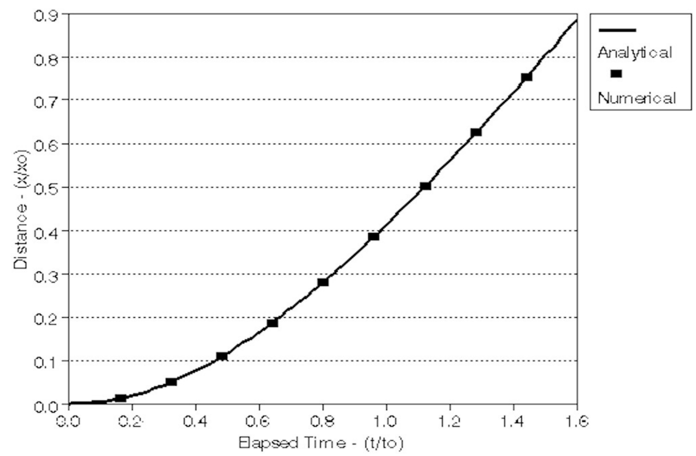
The constant of integration x_c in the above equation is for initial location at zero time. The location x of the body in motion is now given as a function of time t.

To compare Eq. (57) with numerical calculations, both constants of integration x_c and α are set to zero; these values are chosen such that the body starts from a rest position at the origin of the coordinates. To make the analysis unitless, a characteristic time t_0 can be determined from the force per unit mass as $t_0 = c/f$. A characteristic distance x_0 then becomes $x_0 = ct_0$. Using these constants of integration and characteristic dimensions, Eq. (57) becomes

$$\frac{x}{x_0} = \left[1 + \left(\frac{t}{t_0}\right)^2\right]^{1/2} - 1 \qquad (58)$$

The above equation plots a hyperbola in the x,t plane. As it should, the equation equals zero at time t equal to zero. For times much larger than the characteristic time t_0, Eq. (58) approaches a linear curve as the body speed approaches that of light.

Fig. 2 compares Eq. (58) with numerical calculations simulating the same conditions. Numerical calculations are a direct match with analytical data. The characteristic time ratio t/t_0 corresponding to elapsed time is plotted on the abscissa in the figure; the characteristic distance ratio x/x_0 corresponding to distance traveled is plotted on the ordinate.



In the numerical calculations, a body with constant rest mass was subjected to a constant force over incremental time periods until a desired time had elapsed. This time period was measured in the integration reference frame. Dependent integration functions (g_i's of Eq. (1)) were location and momentum. The constant force was set equal to the momentum derivative per Eq. (3). The derivative of location (i.e., velocity) was calculated from momentum using Eq. (4). The program code which follows gives a listing of the derivative function call used in the numerical integration. The function is written in the C programming language ANSI version.

```
 #define R  0            /* location array index */
 #define P  1            /* momentum array index */
 #define C  2.9979E+08  /* speed of light (m/s) */
 #define F  2.9979E+08  /* constant force (N) */
 #define M  1.0          /* body mass (kg) */

 void deriv(double *f, double *g, double t)
/*
 Purpose: - calculates derivatives for Runge-Kutta
            integration algorithm
          - supplies derivatives for body motion
            under constant force
 Input:   g    - dependent integration function
                 array @ time t
          g[R] = location (m)
          g[P] = momentum (kg*m/s)
          t    - time (s)
 Output:  f    - derivative function array @
                 time t
          f[R] = velocity (m/s)
          f[P] = force (N)
*/
 {
  f[R]=g[P]/sqrt(M*M+g[P]*g[P]/C/C); /* compute velocity */
  f[P]=F;                             /* copy force */
  return;
 }
```

B. Photon Rocket with Variable Mass

The one-dimensional acceleration problem of a photon rocket has been solved whereby the rocket expels radiation for thrust to obtain a specified speed[2]. The analytical method uses conservation of momentum and energy to derive an equation relating velocity achieved to fractional mass remaining after propulsion fuel has been expended.

Upon reaching a certain velocity, a photon rocket will have expended a specific amount of radiation in the form of energy taken from the rest mass of the rocket at launch. In addition, this radiation will have a momentum equal to the momentum of the rocket as viewed in the initial rest frame of the rocket at launch. Using conservation of momentum and energy, the state of the rocket at a certain speed can be specified by two equations

$$0 = \gamma fmv - E_r/c \tag{59}$$

In the above equations, the variable *m* represents the initial rest mass of the rocket before launch; *f* indicates the fraction of mass remaining from the initial mass after rocket acceleration. The variable *v* is the final velocity of the rocket after acceleration relative to the at-launch reference frame; *gamma* is a function of the rocket velocity *v* defined by Eq. (10). The variable E_r is the photon radiation expended during the acceleration. Eq. (59) states that the rest energy of the rocket at launch equals the inertial energy of the rocket after acceleration plus radiation energy expelled. Eq. (60) states that the rocket momentum equals the radiation momentum in the opposite direction.

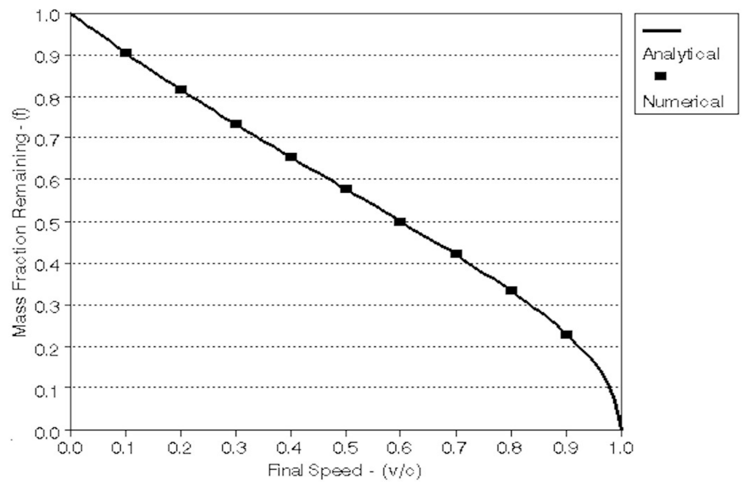
Solving Eq. (60) for radiation energy and substituting this expression into Eq. (59), the result is

$$mc^2 = \gamma fmc^2 + \gamma fmvc \qquad (60)$$

The above equation can be solved for fractional mass in terms of velocity which is

$$f = \frac{(1 - v/c)^{1/2}}{(1 + v/c)^{1/2}} \qquad (61)$$

Eq. (62) shows that, at velocity equal to zero, the fractional mass is unity (no acceleration occurs), and that, as velocity approaches the speed of light, fractional mass goes to zero. Fig. 3 compares Eq. (62) with numerical calculations simulating the same conditions. Numerical and analytical data agree well, and the data agree over a wide range of mass loss rates used in the computer analysis. Speed *v/c* reached by the rocket is plotted on the abscissa in the figure; fractional mass *f* remaining in the rocket after acceleration is plotted on the ordinate.



In the numerical calculations, a body was subjected to a constant mass ejection rate with no other external forces over incremental time periods until a desired speed was reached. The body had an initial mass of one kilogram. Apparent force on the rocket due to ejected mass completely converted to radiation was determined by Eq. (51) in a one dimensional form. The emitted radiation was given the velocity of the speed of light *c* in the negative *x*-axis direction. Mass loss rate was transformed to the at-launch reference frame using Eq. (54). The *gamma* factor for mass loss rate was determined through Eq. (5). The variables were then integrated as ordinary differential equations for mass and momentum which were the dependent integration functions g_i's of Eq. (1). Derivatives were supplied as the mass loss rate and apparent force. Eq. (4) determined rocket velocity from momentum. The program code which follows gives a listing of the derivative function call used in the numerical integration. The function is written in the C programming language ANSI version.

```
 #define P   0           /* momentum array index */
 #define M   1           /* mass array index */
 #define C   2.9979E+08  /* speed of light (m/s) */
 #define DM -2.0E-05     /* mass loss rate (kg/s) */

 void deriv(double *f, double *g, double t)
/*
 Purpose: - calculates derivatives for Runge-Kutta
            integration algorithm
          - supplies derivatives for photon rocket problem
```

```
 Input:   g    - dependent integration function
                 array @ time t
          g[P] = momentum (kg*m/s)
          g[M] = mass (kg)
          t    - time (s)
 Output:  f    - derivative function array @ time t
          f[P] = force (N)
          f[M] = mass time rate of change (kg/s)
*/
 {
  double gamma,             /* gamma factor */
         speed;             /* body speed (m/s) */
  gamma=sqrt(1.0+           /* compute gamma factor */
   g[P]*g[P]/g[M]/g[M]/C/C);
  speed=g[P]/g[M]/gamma;    /* compute speed */
  f[P]=-DM*C+DM*speed;      /* compute apparent force */
  f[M]=DM/gamma;            /* transform mass derivative */
  return;
 }
```

C. Body Motion in a Magnetic Field

The two-dimensional circular motion of a charged body in a constant magnetic field has been developed relativistically[2]. The radius of the motion curvature can be expressed analytically in terms of the magnetic field strength, body charge, and body momentum. The magnetic force ***F*** on a charged body is given by the cross product equation (63)

$$F = kq\,(v \times B) \tag{63}$$

In the above equation, the vector ***v*** is the velocity of the body; the vector ***B*** is the magnetic field. The body in motion has a charge *q*. The variable *k* is a constant of proportionality which in MKS units is unity. For the two-dimensional case of circular motion, the magnetic field is perpendicular to body velocity, and the body travels in a plane that the field is normal to. Consequently, the vector cross product of the velocity and magnetic field in Eq. (63) becomes a direct product; the force vector is directed inward toward the center of the path always perpendicular to the velocity. In a short time period *delta t*, the body experiences a transverse change in momentum *delta* ***p***

$$\Delta p = F\,\Delta t = kqvB\,\Delta t \tag{64}$$

The momentum vector of the moving body is constant in magnitude but rotates with time just as the location vector; the angle change *delta* Θ of the momentum vector is then

$$\Delta\theta = \frac{\Delta p}{p} = \frac{kqvB \quad \Delta t}{p} \tag{65}$$

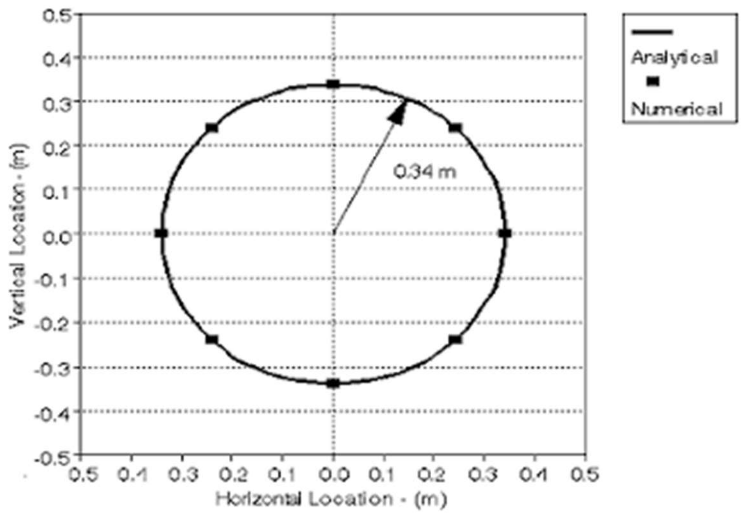
In the same time period *delta t*, the location vector experiences the same change in angle *delta* Θ when the location vector sweeps through an arc of radius *R*

$$\Delta\theta = \frac{\Delta R}{R} = \frac{v \ \Delta t}{R} \tag{66}$$

Combining Eq. (65) and Eq. (66) (thereby eliminating *delta* Θ), the radius of circular motion can be expressed as

$$R = \frac{p}{kqB} \tag{67}$$

The above equation is equivalent to the Newtonian solution; however, the momentum carries the relativistic *gamma* term along with its rest mass and velocity product (Eq. (17)).



For a comparative example, Fig. 4 graphs the circular path of a charged pion under a magnetic field strength of 1.7 Tesla whose direction is normal to the plot. The speed of the pion is 77.9% the speed of light. Per Eq. (67), the radius of the circular motion is 0.34 meters (m). The figure also shows points of location of a pion in motion using numerical integration with the same conditions. Numerical and analytical data agree based on the input parameters selected. Horizontal location of the pion path is plotted on the abscissa in the figure in units of meters; vertical location is plotted on the ordinate using the same units of measure.

In the numerical calculations, a body was subjected to a constant magnetic field over incremental time periods until one period of revolution was achieved. The four dependent integration functions (g_i's of Eq. (1)) were the two-dimensional location and momentum vectors. The particle was initially placed a distance from the origin equal to the orbit radius (Eq. (67)). The initial momentum was directly calculated from the input parameters (Eq. (17)), and the momentum direction was set perpendicular to the location vector. In the integration algorithm, the derivative of location (i.e., velocity) was computed from momentum using Eq. (4). The two-dimensional force vector (Eq. (63)) was set equal to the momentum derivative per Eq. (3). The program code which follows gives a listing of the derivative function call used in the numerical integration. The function is written in the C programming language ANSI version.

```
 #define RX 0          /* location x-axis array index */
 #define RY 1          /* location y-axis array index */
 #define PX 2          /* momentum x-axis array index */
 #define PY 3          /* momentum y-axis array index */
 #define C  2.9979E+08 /* speed of light (m/s) */
 #define B  1.70       /* magnetic field (T) */
 #define Q  1.6022E-19 /* pion charge (C) */
 #define M  2.4887E-28 /* pion rest mass (kg) */

 void deriv(double *f, double *g, double t)
/*
 Purpose: - calculates derivatives for Runge-Kutta
            integration algorithm
          - supplies derivatives for charged particle
            in magnetic field
 Input:   g     - dependent integration function
                  array @ time t
          g[RX] = x-axis location (m)
```

```
          g[RY] = y-axis location (m)
          g[PX] = x-axis momentum (kg*m/s)
          g[PY] = y-axis momentum (kg*m/s)
          t     - time (s)
 Output:  f     - derivative function array @ time t
          f[RX] = x-axis velocity (m/s)
          f[RY] = y-axis velocity (m/s)
          f[PX] = x-axis force (N)
          f[PY] = y-axis force (N)
*/
 {
  double mf;                    /* mass factor (kg) */
  mf=sqrt(M*M                   /* compute mass factor */
   +(g[PX]*g[PX]+g[PY]*g[PY])/C/C);
  f[RX]= g[PX]/mf;              /* compute speed */
  f[RY]= g[PY]/mf;
  f[PX]= f[RY]*Q*B;             /* compute force */
  f[PY]=-f[RX]*Q*B;
  return;
 }
```

CONCLUSION

This article has applied numerical integration to the modeling of body motion in special relativity and illustrated the technique with several examples. With the integration scheme outlined in Sec. II, this formulation can be adapted to many situations which have lent themselves to Newtonian theory. This integration technique introduces computer simulations to the relativistic domain of body motion in dynamics.

Many applications may be possible. In an educational setting, computer code can be developed which models time dilation, length contraction, the twin paradox, and other characteristics of special relativity. Futuristic interstellar space flight can be modeled with the technique. Relativistic rocket propulsion was considered as an example in Sec. V B. However, for this application, gravity is not defined in special relativity and must be approximated if included. Although the technique remains rooted in the classical realm, quantum mechanical approximations may be useful. For example, trajectories of subatomic particles in particle accelerator collisions may be approximated with the technique. A simplified case of a pion in a magnetic field was considered as an example in Sec. V C. With the aid of a numerical integration package, the calculations in implementing the technique are not lengthy, and are accessible to anyone with a personal computer.

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FIGURE CAPTIONS

Figure 1. Location difference transformation reference frames.

Figure 2. Analytical and numerical data comparison of body motion under a constant force.

Figure 3. Analytical and numerical data comparison of the photon rocket problem.

Figure 4. Analytical and numerical data comparison of body motion in a magnetic field.