Special Relativity Elucidates Maxwell's Equations

Introduction

In 1861, James Clerk Maxwell (1831 – 1879) consolidated knowledge gained through experimentation & study of electromagnetism (EM) into a set of equations which bear his name (ref [1]). These equations set up a technique to quantify EM phenomena completely. Once EM characteristics were fully described, the stage was set for further discoveries. At the dawn of the 20th century, Albert Einstein (1879 – 1955) (ref [2]) proposed Special Relativity (ref [3]) that explained many EM issues & Max Planck (1858 – 1947) (ref [4]) published his Quantum solution which set a path for the frontiers of Physics into the 21st century (ref [5]).

Lorentz's Luminiferous Aether: In 1899, Hendrik Lorentz (1853 – 1928*)* (ref [6]) derived his Lorentz Transformation Equations as they are given now complete with (β & γ) designations. However, Lorentz still upheld a universal reference frame for distance & time. When one traveled @ speeds close to (c) from this "preferred" frame, moving matter contracted due to traversing a "luminiferous aether" that pervaded all vacuum. Lorentz also outlined Δ-time by measure of light beams between mirrors. In turn, the aether would cause retardation of light travel time & distance to measure a constant (c).

Lorentz indicated his short comings (ref [6]) citing his incorporation of "a fortuitous compensation of opposing effects". Advances in Science occur many times in independent pairs (ref [7]): Calculus by Isaac Newton & Gottfried Leibniz, Evolution by Charles Darwin & Alfred Wallace, now Relativity by Albert Einstein & Hendrik Lorentz (almost).

What is Special Relativity: In 1905, Albert Einstein proposed Special Relativity (SR) which defined space & time per the Lorentz equations of transformation. These equations modify Newtonian physics for the primary SR postulates (ref [8]):

1) Speed of light in a vacuum (c) is measured the same regardless of its source.
2) All laws of nature are independent of a reference frame's constant velocity ($\boldsymbol{v}$).

Coming into the turn of the last century (1879), experimental results showed that the speed of light was independent of the Earth's orbital motion (ref [9]). In addition, Einstein was conducting "thought experiments." From his mind exercises, Einstein had deemed it unrealistic to travel with a light beam (ref [10]).

Einstein had also determined Isaac Newton ***incorrect*** in describing aspects of time. Newton assumed: "All time flows unchangingly at the same pace, in the same increments, from the infinite past into the infinite future." (refs [11] & [12])

BTW, Einstein proposed General Relativity (GR) in 1915 (ref [13]). In drawing similar analogies to SR, GR encompasses both EM & gravity. When the Universal Gravitational Constant (G) appears in an equation along with the light speed (c), GR assumptions & knowledge are usually being applied. The values of these fundamental physical constants are from reference [14].

		Relativity Constants of Identification		
sym	value	units	label	theory
c	299792458	m/s	speed of light (exact)	SR
G	6.6743E-11	N▪m²/kg²	gravitational constant	GR / SR

If (G) is absent, but (c) is present, then SR assumptions usually apply.

The Lorentz Transformation: In vector equation form for (ct, $\boldsymbol{r}$), space-time components transform to (ct', $\boldsymbol{r'}$) in a reference frame with velocity ($\boldsymbol{v}$), the Lorentz transformation is (ref [15]):

$$ct' = \gamma_v\,[ct - (\boldsymbol{r} \bullet \boldsymbol{\beta}_v)] \qquad \boldsymbol{r}' = \boldsymbol{r} + \gamma_v\,\boldsymbol{\beta}_v\,[(\boldsymbol{r} \bullet \boldsymbol{\beta}_v)\,\sigma_v - ct]$$

assign ($r_{ct} \equiv ct$) then, the space-time 4-vector is (r_{ct}, $\boldsymbol{r}$):

$$(\boldsymbol{r}_{ct})' = \gamma_v\,[r_{ct} - (\boldsymbol{r} \bullet \boldsymbol{\beta}_v)] \qquad \boldsymbol{r}' = \boldsymbol{r} + \gamma_v\,\boldsymbol{\beta}_v\,[(\boldsymbol{r} \bullet \boldsymbol{\beta}_v)\,\sigma_v - r_{ct}]$$

With auxiliary variables:

$$\boldsymbol{\beta}_v \equiv \boldsymbol{v}/c \qquad \gamma_v \equiv (1 - v^2/c^2)^{-1/2} = (1 - \beta_v^2)^{-1/2}$$

$$\sigma_v \equiv [1 - 1/\gamma_v](1/\beta_v^2) = 1/[1+1/\gamma_v]$$

The Lorentz transformation preserves the (ct, $\boldsymbol{r}$) 4-vector length. The transformation equations also show the application of the space-contraction term (σ_v) ($0.5 \le \sigma_v \le 1.0$) (ref [16]).

$$(\boldsymbol{r}')^2 - (ct')^2 = \boldsymbol{r}^2 - (ct)^2 = \text{constant}$$

$$(\boldsymbol{r}')^2 - (r_{ct}')^2 = \boldsymbol{r}^2 - (r_{ct})^2 = \text{constant}$$

In the 4-vector lengths of this article, the [-+++] Minkowski metric (ref [17]) is used with (raised) contravariant numeric indices. The minus sign for the time coordinate indicates that the time-dimension is treated differently from space (x, y, z) dimensions. We can choose a spatial direction at will, but we can only modify our forward speed @ which we inevitably progress one way through the dimension of time. The Lorentz transformation matrix for a given velocity ($\boldsymbol{v}$) that maintains a 4-vector Minkowski length is then:

$$\begin{vmatrix} r'_{ct} \\ r'_x \\ r'_y \\ r'_z \end{vmatrix} = [L_v] \begin{vmatrix} r_{ct} \\ r_x \\ r_y \\ r_z \end{vmatrix} \qquad L_v = \begin{vmatrix} \gamma_v & -\gamma_v\beta_x & -\gamma_v\beta_y & -\gamma_v\beta_z \\ -\gamma_v\beta_x & 1+\gamma_v\sigma_v\beta_x^2 & \gamma_v\beta_x\beta_y & \gamma_v\beta_x\beta_z \\ -\gamma_v\beta_y & \gamma_v\beta_x\beta_y & 1+\gamma_v\sigma_v\beta_x^2 & \gamma_v\beta_y\beta_z \\ -\gamma_v\beta_z & \gamma_v\beta_x\beta_z & \gamma_v\beta_y\beta_z & 1+\gamma_v\sigma_v\beta_z^2 \end{vmatrix}$$

The 4×4 matrix (L_v) is a velocity ($\boldsymbol{v}$) boost transformation matrix for 4-vector transformations (ref [18]). With matrix indices given standard as ($a_{ij} = a_{row,column}$) matrix algebra is reviewed in reference [19]. The matrix algebra of (L_v) duplicates the vector algebra of the Lorentz transformation equations.

EM Support Vectors

Other scalars & vectors are supported in EM / SR to describe physical systems, including force & acceleration, work & energy. These definitions are outlined below.

Force & Acceleration: Borrowing from Isaac Newton (1642 – 1726) & his inverse-square force law of gravity between two bodies of mass ($m_1(\boldsymbol{x}_1)$ & $m_2(\boldsymbol{x}_2)$) (ref [20]), Charles-Augustin de Coulomb (1736 – 1806) (refs [21] & [22]) defined & measured the electrostatic force between two charges ($q_1(\boldsymbol{x}_1)$ & $q_2(\boldsymbol{x}_2)$) in 1785 (ref [23]). The force is between charges ($q_1(\boldsymbol{x}_1)$) @ vector location ($\boldsymbol{x}_1$) & ($q_2(\boldsymbol{x}_2)$) @ vector location ($\boldsymbol{x}_2$) separated by a distance ($\boldsymbol{r}_{21} = \boldsymbol{x}_2 - \boldsymbol{x}_1$). Monsieur Coulomb was a Frenchman, so his name is pronounced “COO-lam” (ref [24]).

$$\boldsymbol{F}_{\text{Newton}} = [G][m_1 m_2/(\mathbf{r}_{12} \bullet \mathbf{r}_{12})]\,\hat{\mathbf{n}}_{12} \quad \Rightarrow \quad \boldsymbol{F}_{\text{Coulomb}} = [1/(4\,\pi\,\varepsilon_0)][q_1 q_2/(\mathbf{r}_{12} \bullet \mathbf{r}_{12})]\,\hat{\mathbf{n}}_{12}$$

A force ($\boldsymbol{F}_{obj}$) is detected if an acceleration ($\boldsymbol{a}_{obj}$) of an object (obj) with mass (m_{obj}) is observed. From Newton’s calculus, acceleration is the time-derivative of velocity ($\boldsymbol{v}_{obj}$) which in turn is the time-derivative of location ($\boldsymbol{x}_{obj}$). Then, Newton’s 2nd Law of motion is (ref [25]):

$$\boldsymbol{F}_{\text{obj}} = m\boldsymbol{a}_{\text{obj}} \qquad \text{where} \qquad \boldsymbol{a}_{\text{obj}}(t) \equiv d/dt\ \boldsymbol{v}_{obj}(t) \equiv d/dt\ [\ d/dt\ \boldsymbol{x}_{obj}(t)\ ]$$

If the object moves with no acceleration but constant velocity, then the object can define a Galilean inertial reference frame (ref [26]). An important note, the Earth provides almost a Galilean inertial reference frame. However, a “Coriolis force” can be detected due to the rotation of the Earth (ref [27]). Other Earth accelerations (from orbit around Sun & orbit around the Milky Way) are much weaker.

In convincing 17th century scholars, we are on a moving Earth, Galileo Galilei (1564 – 1642) remarked, “When you drop a rock from the mast of a moving ship, the rock lands at the base of the mast!” (ref [28]), On the other hand, our latitudinal trade wind patterns are affected significantly due to the Earth’s rotation (ref [29]) as the winds move across its surface.

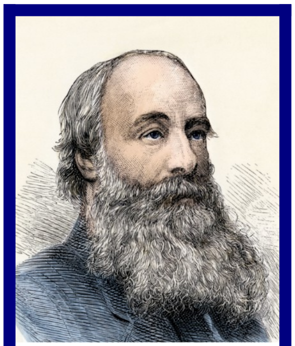
Energy & Work: A physicist’s definition of energy (E) & “stored work” is numerically well defined / evaluated. This definition may be different from what economists & others may define as work. Once a local force field ($\boldsymbol{F}(\boldsymbol{x})$) is defined across space, the definition of work is also (ref [30]):

$$\text{Work} \equiv \int \boldsymbol{F}(\boldsymbol{x}) \bullet d\ell = -\,\Delta E_P = -[E_P(\boldsymbol{x}_2) - E_P(\boldsymbol{x}_1)]$$

For this application, work is relative to the points ($\boldsymbol{x}_1$ & $\boldsymbol{x}_2$). Also, the work integrand is a dot product. Then, if the path of an object is perpendicular to the force ($\boldsymbol{F}(\boldsymbol{x})$), then ***no*** work is performed. When work is positive, the force / path dot product is positive & the force from the field performs work releasing potential energy ($\Delta E_P < 0$). When work is negative, potential energy ($\Delta E_P > 0$) is stored in the field & the force has opposed the path of motion (ref [31]).

The metric measure of energy is “Joule” named after James Prescott Joule (ref [32]).

Mister Joule was an English brewer (ref [33]) & his name sounds like "JOOL" (ref [34]). A Joule (J) unit of energy is obtained by moving an object one meter against a constant force of one Newton. When a Coulomb count of like charges is moved across a meter against an EM force of one Newton, that's the definition of a Volt which is a Joule per Coulomb, a measure of energy per charge (ref [35]).



James P Joule
1818 – 1889

The public is rarely exposed to the term "Joule" (J), using anything but this standard metric unit of energy (ref [36]). See the table below. The utility companies use kiloWatt-hours (ref [37]); architects use British Thermal Units (BTU's) (ref [38]); the US Food & Drug Administration (FDA) uses calories (ref [39]), air conditioning manufacturers use tons of ice (ref [40]) … anything to make life confusing!

Units of Energy

energy units	abr	conversion to metric Joules	application	weight lifter exercise energy
Joules	J	1.00000 J	scientific	2352.00 J
Calorie	Cal	4.18400 J	food	562.141 Cal
British Thermal Unit	BTU	1.05506 kJ	heating	2.22927 BTU
KiloWatt-Hour	kW▪hr	3.60000 MJ	electrical utilities	0.00065 kW▪hr
Refrigeration Tons	RT	12.66067 MJ	air conditioning	0.00019 RT

Exercising While Working: In the Earth's gravitational field near its surface, the "g-force" gravitational acceleration constant (about 9.8 m/s² or 32 ft/s²) can readily assess work being performed. In a standard metric weight lifting gym, the bar @ the bench press (ref [41]) has 20 kilograms (kg) of mass; each large plate is 20 kg each. Then, for a standard bench "warm-up" of 10 reps @ 60 kg of mass (about 132 pounds weight) for a nominal arm reach of about 0.4 meters (about 16 inches), the work done by the fitness enthusiast is:

$$Work_{\text{lifter}} = -n_{rep}\,(mg \cdot \Delta y)$$

$$m_{total} = \left[2*\left(\frac{20\,kg}{plate}\right)+\left(\frac{20\,kg}{\text{bar}}\right)\right] = 60\,kg$$

$$Work_{lifter} = \left(\frac{10\,reps}{set}\right)\left(\frac{9.8\,m}{s^2}\right)\left(\frac{60\,kg*0.4\,m}{rep}\right)\left(\frac{1\,J\,s^2}{1\,kg\,m^2}\right)\left(\frac{1\,kJ}{1000\,J}\right) = 2.352\,kJ$$

From the point of his / her medical doctor, the fitness enthusiast is doing something worthwhile in his / her spare time, ***exercising***. From the standpoint of an economist, the fitness enthusiast is contributing ***no*** work to the US Gross National Product (GNP) by exercising @ a gym. To the physicist / engineer, the fitness enthusiast has most definitely performed about ***2.3 kiloJoules*** of work. It's all how one defines ***work***!

Lorentz Force Law: The ***E*** & ***B*** fields of Maxwell's Equations denote 3D electric & magnetic vector fields respectively defined for a spatial location @ a particular time in a particular environment. These field 3D vectors are essentially 3 separate component functions specifying a unique direction & magnitude @ each applicable location of interest. The fields are derived from existing scalar EM charges in a given locale along with the vector motion of these charged particles.

A scalar charge of a particle is a single number describing a property constant of the particle. An introduced charge moving through the defined fields experiences a force as given by Lorentz Force Law (ref [42]):

$$\boldsymbol{F}_t(\boldsymbol{r}_t) = q_t[\boldsymbol{E}(\boldsymbol{r}_t) + \boldsymbol{v}_t \times \boldsymbol{B}(\boldsymbol{r}_t)]$$

This law calculates an EM force ($\boldsymbol{F}_t$) on a particle @ location ($\boldsymbol{r}_t$) with a charge (q_t) traveling with velocity ($\boldsymbol{v}_t$) experiencing an electric field ($\boldsymbol{E}$) & magnetic field ($\boldsymbol{B}$). When Hendrik Lorentz proposed his namesake equation in 1895, magnetism seemed an inexplicable force. However, in 1905, Albert Einstein showed ***B*** fields to be necessary to explain a finite light speed. Furthermore, Maxwell's equations, in quantifying magnetism, were correct & exact.

Conservation of Charge: In beta decay, an isolated neutron splits into a single pair of a stable oppositely charged proton & electron in about 15 minutes on average. At the end of some stars' lives, that are much larger than our Sun, the opposite occurs. Their fuel is depleted; their masses collapse to the extent that gravity compresses electrons into atomic nuclei; proton & electron charged pairs combine forming neutrons in a neutron star (ref [43]).

However, for physical reactions on Earth, total electric charge remains constant in almost all processes. A differential form of conservation of charge is (ref [44]):

$$(\nabla \cdot \boldsymbol{J}) + \frac{\partial \rho}{\partial t} = 0$$

The equation can be stated as current density flow exiting a volume plus the volumetric charge density increase are zero. Using a 4-vector notation ($J^o \equiv c\rho$ & $r^o \equiv ct$):

$$(\nabla \cdot \boldsymbol{J}) + \frac{\partial (c\rho)}{\partial (ct)} = 0 \quad \Rightarrow \quad (\nabla \cdot \boldsymbol{J}) + \frac{\partial J^0}{\partial r^0} = 0 \quad \Rightarrow \quad \partial^\alpha J^\alpha = 0$$

By convention, if coordinate indexes range across 3D space, Latin letters are used. If a 4-vector component is being represented, Greek letters are used. For (raised) contra-variant indices, Greek letters that appear twice in a matrix equation are summed over the applicable dimensions $x^\alpha \in (ct, x, y, z)$.

MKSA Units of Measure: MKSA (meter / kilogram / second / amp) units are contained in the EM conversion constants, (ε_0) & (μ_0), cited below. Electric permittivity (ε_0) relates electric charge densities (ρ) & electric (***E***) fields. The magnetic permeability (μ_0) relates electric current densities (***J***) & magnetic (***B***) fields (ref [14]) in a vacuum. Due to EM properties:

$$\varepsilon_0 \, \mu_0 = 1/c^2$$

The form of Maxwell's Equations discussed herein use MKSA in a vacuum, exclusively. Historical dipole approximations of Maxwell's Equations within dielectric & magnetic media are given here (eduDipole.pdf).

Maxwell's Equations

The four Maxwell's Equations are heavy into calculus notations (ref [45]) that ***cannot*** be described as simple Greek letters. The symbols to be encountered fall within the realm of multi-variable calculus (ref [46]). However, given a well-defined set of static & moving charge distributions, the four equations give requirements to formulate ***E*** & ***B*** fields to evaluate the motion of "test" charges moving within the fields.

Maxwell's Equations can be put in SR 4-vector form also. Without further a due, the four equations are (refs [47] & [48] & [49]):

	3-vector form	4-vector form
1)	$\nabla \bullet \boldsymbol{E} = \rho/\varepsilon_0$	$\nabla \bullet \boldsymbol{E}/c = \mu_0 J_{ct}$
2)	$\nabla \bullet \boldsymbol{B} = 0$	$\nabla \bullet \boldsymbol{B} = 0$
3)	$\nabla \times \boldsymbol{E} + \partial \boldsymbol{B}/\partial t = 0$	$\nabla \times \boldsymbol{E}/c + \partial_{ct} \boldsymbol{B} = 0$
4)	$\nabla \times \boldsymbol{B} - (1/c^2)(\partial \boldsymbol{E}/\partial t) = \mu_0 \boldsymbol{J}$	$\nabla \times \boldsymbol{B} - (\partial_{ct} \boldsymbol{E})/c = \mu_0 \boldsymbol{J}$

The four ***new*** symbols (ref [50]) in the above equations are the symbols (∂) ($\int$) ($\nabla\bullet$) ($\nabla\times$).

Calculus Symbols (∂) ($\int$) (∇) ($\nabla\bullet$) ($\nabla\times$): When several independent variables are in a function, the ***partial derivative*** term ($\partial/\partial t$) indicates that the derivative should be performed with the independent denominator time (t), keeping other independent variables (spatial x, y, z) constant (ref [51]). In Maxwell's Equations, the partial with respect to (wrt) time (t) indicates the last two Maxwell Equations relate ***E*** or ***B*** field changes in time to ***B*** or ***E*** field changes in space, respectively.

When an integration ($\int$) is performed, a function is found whose derivative equals the integrand (anti-differentiation). Finding the ***integral*** is also a summation of the integrand over the area of interest (ref [52]).

The ***gradient* (∇)** of a potential is a vector operator on a scalar & measures the spatial change or "steepness" of the potential @ location (***x***) (ref [53]). An analogy of flowing water downhill is applicable (ref [54]). A river flows slowly near a ***flat*** coastal plane with little slope verses a fast waterfall on a ***steep*** mountain side with great slope. If the potential's gradient is "very steep" (@ $\boldsymbol{x}_{steep}$), then the resultant electric field will be strong because charge density is high near ($\boldsymbol{x}_{steep}$). A test charge ($q_t(\boldsymbol{x}_{steep})$) will experience a greater force ($\boldsymbol{F}(\boldsymbol{x}_{steep}) = q_t\, \boldsymbol{E}(\boldsymbol{x}_{steep})$) & accelerate faster ($\boldsymbol{a}_t(\boldsymbol{x}_{steep}) = \boldsymbol{F}(\boldsymbol{x}_{steep})/m_t$).

The dot product ***divergence*** (∇•) term performs math calculations on a vector field that give the slope or gradient of a 3-Dimensional field in space (ref [55]). This operation results in a scalar that denotes sources or sinks of spatial change in a vector field.

The cross product ***curl*** (∇×) term of a vector field performs matrix calculus on a vector field to estimate the location's infinitesimal rotation (ref [56]). This operation results in a vector that denotes the rotation axis & magnitude in a vector field. The curl of a vector measures for each dimensional component the spatial change or "steepness" of the potential (***A***(***x***)) @ location (***x***). In addition, because a matrix cross product is incorporated into the curl definition, the slope or "steepness" is expressed in a vector @ a right angle to the averaged current density flow (***J***(***x***)) incorporated in the (***A***(***x***)) evaluation.

Φ(x) - Electric Scalar Potential: The ***E*** field is based on the electric potential of the charge configuration at hand. The Electric Potential (Φ(***x***)) is essentially the summation of stored energy of the source charges causing the ***E*** field (ref [18]). Then, the integral potential whose derivative is the ***E*** field between charges (q_1 & q_2) are:

$$\boldsymbol{F}_{Coulomb} = \left(\frac{q_1 q_2}{4\pi\epsilon_0}\right)\left(\frac{(\boldsymbol{x_2}-\boldsymbol{x_1})}{|\boldsymbol{x_2}-\boldsymbol{x_1}|^{3/2}}\right) = -\nabla\left(\frac{q_1 q_2}{4\pi\epsilon_0}\right)\left(\frac{1}{|\boldsymbol{x_2}-\boldsymbol{x_1}|^{1/2}}\right)$$

The electric potential (Φ($\boldsymbol{x}_2$)) is:

$$\Phi(\boldsymbol{x_2}) = \left(\frac{q(\boldsymbol{x_1})}{4\pi\epsilon_0}\right)\left(\frac{1}{|\boldsymbol{x_2}-\boldsymbol{x_1}|}\right)$$

The work required to assemble a group of charges is given by the summations:

$$Work = \frac{1}{4\pi\epsilon_0}\sum_{i=1}\sum_{j=1}\frac{q_i(\boldsymbol{x_i})q_j(\boldsymbol{x_j})}{|\boldsymbol{x_i}-\boldsymbol{x_j}|}$$

If discrete summations are transformed into continuous integrals then:

$$Work = \frac{1}{4\pi\epsilon_0}\int_V\int_{V'}\frac{\rho(\boldsymbol{x})\rho(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}d^3\boldsymbol{x}'d^3\boldsymbol{x} \quad ⇨ \quad \Phi(\boldsymbol{x}) \equiv \frac{1}{4\pi\epsilon_0}\int_{V'}\frac{\rho(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}d^3\boldsymbol{x}'$$

Substituting,

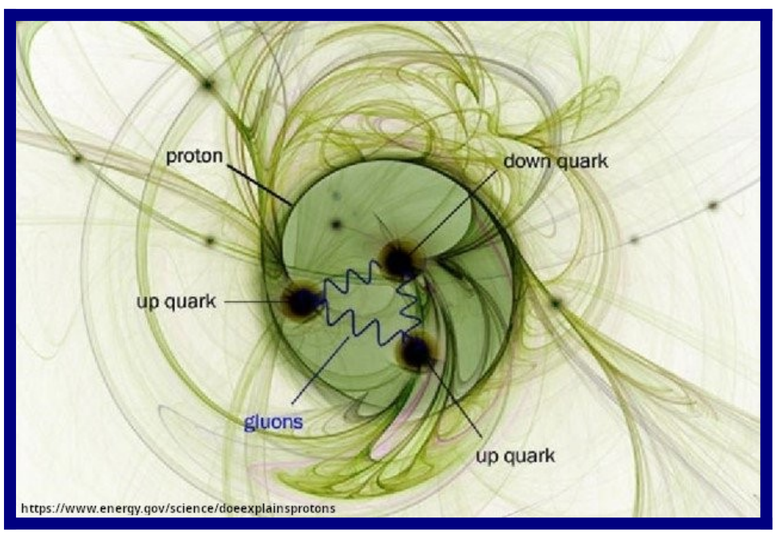
$$Work = \iiint \rho(\boldsymbol{x})\Phi(\boldsymbol{x})d^3x \quad ⇨ \quad \boldsymbol{E}(\boldsymbol{x}) = -\nabla\Phi(\boldsymbol{x})$$

The electric potential function ($\Phi(\boldsymbol{x})$) evaluates the stored energy per charge (Volts - Joules/Coulomb) of the EM charge distribution within the volume (V) of interest.

A(x) - Magnetic Vector Potential: From 1st principals, a line current (I) creates a magnetic field ***B,*** through which a nearby "test" charged particle (q_t) will experience a magnetic force (ref [18]). The current is defined as electrons moving through a conductor of almost immobile equally charge protons making the net static charge of a wire carrying current zero. By sign convention (ref [57]), current flows from negative poles to positive poles.

Electron − Proton − Neutron Comparison			
label	**electron**	**proton**	**neutron**
Particle type	Lepton	Hadron	Hadron
Magnetic spin	+½	+½	-½
Charge (zC = 10^{-21} C)	-160.2	160.2	0
Charge (±e)	(-1 e)	(+⅔ e)(+⅔ e)(-⅓ e)	(+⅔ e)(-⅓ e)(-⅓ e)
Quark types	n/a	up up down	up down down
# of Quarks	0	3	3
Mass per $m_{electron}$	1	1836	1839
Radius (fm = 10^{-15} m)	~0 (point)	0.88	0.8

Basic proton (ref [58]) / neutron (ref [59]) / electron (ref [60]) differences are given in the above table. The positively charge protons & neutrally charged neutrons are about equal in mass, much heavier than the electrons & bound in the nucleus of the conductor's atoms. The subatomic particles are simply ***not*** "equal but opposite" charged particles.



The electrons are essentially point particles, but nucleons have a finite radius as depicted to the right (ref [61]). A down quark along with 2 up quarks are shown within the proton held together by gluons.

Negative electrons are very dynamic among near static protons that are 1836 times more massive. Still, the conductor is essentially neutral before & during current flow. A magnetic field occurs because of the perceived length contraction in moving electron density due to SR effects by a stationary observer (ref [62]).

Similar to charges ($q_i(\boldsymbol{x})$) consolidating into charge density ($\rho(\boldsymbol{x})$), moving charges ($q_i\boldsymbol{v}_i$) can be consolidated into current density ($\boldsymbol{J}(\boldsymbol{x})$) (ref [18]).

$$\boldsymbol{A}(\boldsymbol{x}) \equiv \frac{\mu_0}{4\pi}\int_V \frac{\boldsymbol{J}(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}d^3x' \quad \Rightarrow \quad \boldsymbol{B} = \nabla\times\boldsymbol{A}$$

The 3-Dimensional magnetic vector potential ($\boldsymbol{A}(\boldsymbol{x})$) can be considered three scalar potentials for the three components of the current density vector integration. The function ($\boldsymbol{A}(\boldsymbol{x})$) evaluates the stored energy per charge (Volts − Joules/Coulomb) of the EM current distribution in vector form within the volume (V) of interest.

Time Dependent Direct Field Tensors

The ***B*** field expressions are repeated here for reference. To express a time dependent ***E*** field from scalar & vector potentials, from Maxwell's 3rd Law, a time varying ***E*** field is expressed as:

$$\boldsymbol{E} = -\,\partial\boldsymbol{A}/\partial(ct) - \nabla\Phi$$

Then,

$$A_{ct}(\boldsymbol{x},t) = -\left[\Phi(\boldsymbol{x},t) \equiv \frac{\mu_0}{4\pi}\int_V \frac{J_{ct}(\boldsymbol{x}',t')}{|\boldsymbol{x}-\boldsymbol{x}'|}d^3\boldsymbol{x}'\right] \;\&\; \boldsymbol{A}(\boldsymbol{x},t) \equiv \frac{\mu_0}{4\pi}\int_V \frac{\boldsymbol{J}(\boldsymbol{x}',t')}{|\boldsymbol{x}-\boldsymbol{x}'|}d^3x'$$

$$\boldsymbol{E} = -\,[\partial\boldsymbol{A}/\partial r_{ct} - \nabla A_{ct}] \quad \& \quad \boldsymbol{B} = \nabla\times\boldsymbol{A}$$

The retarded time (t') evaluating the 4-vector potential (ref [63]) is given by (ref [64]):

$$t' = t - |\mathbf{r} - \mathbf{r}'|/c$$

The retarded time (t') gives the time when the field actually began to propagate in evaluating the potentials (A^{γ}).

For Einstein's field tensor notation, the following 4-vector notation is used:

$x^{\alpha} \in (ct, x, y, z) = (r_{ct}, r_x, r_y, r_z) = (r_{ct}, \boldsymbol{r})$

$A^{\gamma} \in (A_{ct}, A_x, A_y, A_z) = (-\Phi, \boldsymbol{A})$.

$J^{\gamma} \in (J_{ct}, J_x, J_y, J_z) = (c\rho, \boldsymbol{J})$

$\partial^{\alpha} = \partial/\partial x^{\alpha} \in (\partial/\partial(ct), \partial/\partial x, \partial/\partial y, \partial/\partial z) = (\partial/\partial(ct), \nabla)$.

Equations 1 & 4 as Direct Field Tensors: To express the inhomogeneous Maxwell's Equations using field tensors, charge distributions are expressed using the above index notation. From Maxwell's Equations 1 & 4 with the charge continuity equation:

$$\boldsymbol{E} = -\,[\partial\boldsymbol{A}/\partial r_{ct} - \nabla A_{ct}] \;\Rightarrow\; \begin{array}{l} E_x/c = F^{10} = -\,(F^{01} = \partial_{ct}A_x + \partial_x\Phi) = -\,(F^{01} = \partial_{ct}A_x - \partial_x A_{ct}) \\ E_y/c = F^{20} = -\,(F^{02} = \partial_{ct}A_y + \partial_y\Phi) = -\,(F^{02} = \partial_{ct}A_y - \partial_y A_{ct}) \\ E_z/c = F^{30} = -\,(F^{03} = \partial_{ct}A_z + \partial_z\Phi) = -\,(F^{03} = \partial_{ct}A_z - \partial_z A_{ct}) \end{array}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \quad \Rightarrow \quad \begin{array}{l} F^{32} = -(B_x = F^{23} = \partial_y A_z - \partial_z A_y) \\ B_y = F^{31} = -(F^{13} = \partial_x A_z - \partial_z A_x) \\ F^{21} = -(B_z = F^{12} = \partial_x A_y - \partial_y A_x) \end{array}$$

Coordinate indexes define the field tensor placement ($F^{\alpha\beta}$) from components of Maxwell's Equations 1 & 4. The direct anti-symmetric field strength tensor:

$$F_E^{\alpha\beta} \equiv \partial^\alpha A^\beta - \partial^\beta A^\alpha$$

$$F_E^{\alpha\beta} = \left| \begin{array}{cccc} 0 & \partial_{ct} A_x - \partial_x A_{ct} & \partial_{ct} A_y - \partial_y A_{ct} & \partial_{ct} A_z - \partial_z A_{ct} \\ \partial_x A_{ct} - \partial_{ct} A_x & 0 & \partial_y A_x - \partial_x A_y & \partial_x A_z - \partial_z A_x \\ \partial_y A_{ct} - \partial_{ct} A_y & \partial_x A_y - \partial_y A_x & 0 & \partial_z A_y - \partial_y A_z \\ \partial_z A_{ct} - \partial_{ct} A_z & \partial_z A_x - \partial_x A_z & \partial_y A_z - \partial_z A_y & 0 \end{array} \right|$$

$$F_E^{\alpha\beta} = \left| \begin{array}{cccc} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{array} \right|$$

The above tensor is "anti-symmetric" because it's negative symmetric about its zero-diagonal. The 1st & 4th Maxwell Equations are termed "inhomogeneous" because of the non-zero charge & current density terms on the right side of the equations.

1st)	$\nabla \cdot \mathbf{E}/c = \mu_0(c\rho)$	$\Rightarrow$	$(\partial_x E_x + \partial_y E_y + \partial_z E_z)/c = \mu_0 J_{ct}$
4th)	$\nabla \times \mathbf{B} - \partial_{ct}\mathbf{E} = \mu_0 \mathbf{J}$	$\Rightarrow$	$(\partial_y B_z - \partial_z B_y) - \partial_{ct} E_x / c = \mu_0 J_x$ $(\partial_z B_x - \partial_x B_z) - \partial_{ct} E_y / c = \mu_0 J_y$ $(\partial_x B_y - \partial_y B_x) - \partial_{ct} E_z / c = \mu_0 J_z$

All components of Maxwell's Equations 1 & 4 can be expressed in a compact form:

$$\partial^\alpha F_E^{\beta\gamma} - \partial^\beta F_E^{\alpha\gamma} = \mu_0 J^\gamma$$

Equations 2 & 3 as Inverted Field Tensors: By convention, electron electric & magnetic properties are chosen to be ($q_e = (-e)$) & ($q_m = 0$). Protons then have properties ($q_e = (+e)$) & ($q_m = 0$). Likewise, a valid inverted tensor field is defined with a transformation **E** & **B** field conversion (ref [18]). Here ($q_m = (-e)$) & ($q_e = 0$). This inverted anti-symmetric field strength tensor is:

$$F_E^{\alpha\beta} = \left| \begin{array}{cccc} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{array} \right|$$

$$\mathbf{E}/c \Rightarrow \mathbf{B} \;\;\&\;\; \mathbf{B} \Rightarrow (-\mathbf{E}/c) \quad \Rightarrow \quad (E_i/c) \Rightarrow B_i \;\;\&\;\; B_i \Rightarrow (-E_i/c)$$

Making the required substitutions,

$$F_B^{\alpha\beta} = \begin{vmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & -E_z/c & 0 & E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{vmatrix}$$

The 2nd & 3rd Maxwell Equations are termed "homogeneous" because of the zero scalar on the right side of the equations.

2nd) $\nabla \bullet \boldsymbol{B} = 0$	⇨	$\partial_x B_x + \partial_y B_y + \partial_z B_z = 0$
		$(\partial_y E_z - \partial_z E_y)/c + \partial_{ct} B_x = 0$
3rd) $(\nabla \times \boldsymbol{E})/c + \partial_{ct} \boldsymbol{B} = 0$	⇨	$(\partial_z E_x - \partial_x E_z)/c + \partial_{ct} B_y = 0$
		$(\partial_x E_y - \partial_y E_x)/c + \partial_{ct} B_z = 0$

All components of Maxwell's Equations 2 & 3 can be expressed in a compact form:

$$\partial^\alpha F_B^{\beta\gamma} - \partial^\beta F_B^{\alpha\gamma} = 0$$

MKSA Constants & Metrics for EM: The following data uses MKSA (meter / kilogram / second / amp) units of measure (ref [14]) to describe EM phenomena in a vacuum (si4x6.pdf):

MKSA Derived Units for Electromagnetism					
label	type	sym	alternate units		value
pi	constant	π	–		3.141592653589793
light speed in vacuum	constant	c	m/s (exact)		299792458
electric permittivity	constant	ε_0	Farad/meter	A²/N/(m/s)²	8.8541878128E-12
magnetic permeability	constant	μ_0	Henry/meter	N/A²	$4\pi\times10^{-7}$
			Henry/meter	N/A²	1.2566370621E-06
electric potential	scalar	Φ	Volts	N/A•m/s	–
magnetic potential	vector	$\boldsymbol{A}$	Volts	N/A•m/s	–
electric field	vector	$\boldsymbol{E}$	Volt/meter	N/A/s	–
magnetic field	vector	$\boldsymbol{B}$	Tesla	N/A/m	–
charge density	scalar	ρ	A•s/m³		–
current density	vector	$\boldsymbol{J}$	A/m²		–

Summary

The key to Maxwell's Equations simplification, was Einstein's observation that the light speed-time product (ct) is another dimension of space (x, y, z) in a new 4-vector space-time (ct, x, y, z). The incorporation of time in space-time coordinates of Maxwell's 4 Equation applications is straight forward. Lorentz Law is then applied to a "test" charge @ location ($\boldsymbol{r}_t$) affected by derived $\boldsymbol{E}$ & $\boldsymbol{B}$ fields:

$$\boldsymbol{F}_t(\boldsymbol{r}_t) = q_t[\boldsymbol{E}(\boldsymbol{r}_t) + \boldsymbol{v}_t \times \boldsymbol{B}(\boldsymbol{r}_t)]$$

First, the 4-vector potential describing local 4-vector current densities is formulated:

$$t' = t - |r - r'|/c \quad \& \quad J^\alpha = (c\rho, J_x, J_y, J_z) \quad \& \quad A^\alpha = (-\Phi, A_x, A_y, A_z)$$

$$A_{ct}(\mathbf{x},t) \equiv -\frac{\mu_0 c}{4\pi}\int_V \frac{\rho(\mathbf{x}',t')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \quad \& \quad \mathbf{A}(\mathbf{x},t) \equiv \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{x}',t')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$$

Second, Direct (E_i) & Inverted (B_i) field strength tensors of ***E*** & ***B*** fields are given by the formulas evaluated from the 4-vector potential:

$$F_E^{\alpha\beta} \equiv \partial^\alpha A^\beta - \partial^\beta A^\alpha$$

$$F_E^{\alpha\beta} = \begin{vmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{vmatrix}$$

$$\boldsymbol{E}/c \Rightarrow \boldsymbol{B} \quad \& \quad \boldsymbol{B} \Rightarrow (-\boldsymbol{E}/c)$$

$$F_B^{\alpha\beta} = \begin{vmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & -E_z/c & 0 & E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{vmatrix}$$

The two-step process for evaluation ***E*** & ***B*** fields was derived from the direct (F_E) or inverted (F_B) anti-symmetric field strength tensor in 1st & 4th inhomogeneous or 2nd & 3rd homogeneous empirical Maxwell's Equations. These 4 equations were simplified to the vector component forms:

$$\partial^\alpha F_E^{\beta\gamma} - \partial^\beta F_E^{\alpha\gamma} = \mu_0 J^\gamma \quad \& \quad \partial^\alpha F_B^{\beta\gamma} - \partial^\beta F_B^{\alpha\gamma} = 0$$

Problem Solved!

Two great synthesis of knowledge have occurred in Classical Physics. One synthesis is outlined here, where one looks at the 4 Maxwell's Equations & can't begin to see an underlying theme that Nature is trying to achieve. The key is: 1) clocks run at different rates, 2) time is another dimension (albeit unique) in space-time. Thereby, Nature keeps the measurement of light speed (*c*) constant throughout the Universe. These ideas consolidate Maxwell's Equations. Thank you Albert Einstein!

The second great synthesis occurred at the dawn of Classical Physics, when Johannes Kepler (1571 – 1630) proposed his bizarre 3 Laws of Planetary Motion (ref [65]). With experiments (ref [66]) from Galileo Galilei (1564 – 1642), Isaac Newton (1642 – 1726) invented Calculus (ref [67]) to explain why the planets move in their Kepler described orbits. That amazing accomplishment is covered elsewhere (ref [68]).

Although physicists may say there are many, in my opinion, the present "can of worms" in Physics is Quantum Theory. This field of modern Physics is waiting for an Einstein, someone remarked “it may take two”, to figure out just what the hell Nature is trying to do with all of Quantum Theory’s bizarre rules & laws. In 1912, Albert Einstein observed, “The more success the Quantum Theory has, the sillier it looks" (ref [69]). Nevertheless, Quantum Theory investigates Nature on an atomic scale & has had many, many successes since 1912.

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