Calculus Elucidates Kepler's Laws

Introduction

Like playing a pirated video game without a manual, human kind was put here on Earth, and only due to his / her wits had to figure out what the hell was going on. Living under the stars, humans studying the night sky noticed 5 points of light that did not twinkle like the other stars. They also moved across the darkness unlike the other stars. Different cultures assigned different names to these points of lights, the planets now go by their modern names taken from the ancient Romans: Mercury, Venus, Mars, Jupiter, Saturn (ref [1]). In turn, the Romans adopted their gods from the ancient Greeks across the Adriatic Sea. As we shall find, it's the curiosity of humans about the motion of these planets which unlocked the power of Science in the Renaissance by a spark from Nicolaus Copernicus (ref [2]).

What Our Ancestors Saw as Planets

The 5 visible planets move along a similar arc across the sky. This arc also contains the path of the Sun & Moon reflecting a mutual plane that contains most orbiting bodies of the solar system including Earth. Astronomers call this path the "ecliptic" plane (ref [3]). The planets also have unique visual characteristics used to identify them by sight (ref [4]). The following table links reference [5] derivative subtopics.

Planet Knowledge of the Ancients

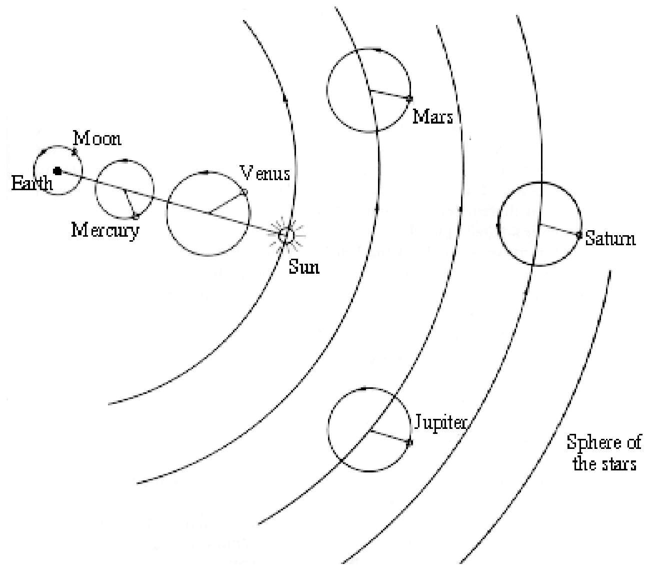
planet	symbol	Greek God	purpose	Period (years)	brightness rank	near Sun	color	with low cost telescope
Mercury	☿	Hermes	Messenger of Gods	0.24	2nd	in glare of sun set / rise	bright yellow dot	–
Venus	♀	Aphrodite	God of Love	0.62	1st	morning or evening star	bright white-blue dot	disc phases visible
Mars	♂	Ares	God of War	1.88	3rd	wanders across sky	bright orange dot	–
Jupiter	♃	Zeus	God of the Sky	11.86	4th	wanders across sky	non-twinkling white dot	some moons visible
Saturn	♄	Cronus	God of the Harvest	29.46	5th	wanders across sky	non-twinkling white dot	rings & some moons visible

The planets can be placed into two groups, Mercury & Venus stay near the Sun in their movements, the other planets wander along the ecliptic detached from the Sun's position.

An Initial Explanation with an Inaccurate Assumption

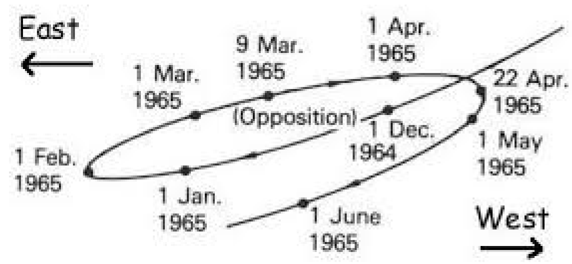
In ancient history, various cultures developed different explanations for the motions of the planets. In Alexandria, Egypt, Claudius Ptolemy (c 100 – c 170 AD) proposed a geocentric, or Earth-centered model of the solar system in *Almagest* (c 150 AD) (ref [6]). Geocentric was assumed because, just as in a chariot or on a boat, a rider in motion can tell if he / she is moving. The Earth seems stationary, therefore the Sun must orbit the Earth.

The great Greek philosopher, Aristotle (384 – 322 BC) had conjectured that the natural state of all physical bodies is to come to rest or be at rest; he had no concept of friction (ref [7]). Indeed, Helios, the god of the Sun in Greek mythology, had a chariot with four horses that "dragged" the Sun across the sky daily (ref [8]).



Ptolemy employed epicycles to explain discrepancies in his theory to match observed data. Epicycles are smaller orbits carried along larger orbits. The figure to the right, depicts Ptolemy's planetary explanation (ref [9]). When added, individual epicycles were given a period of time for the planet to complete to agree with observed data. Thus, the motions or the inner planets, seeming to hang around the Sun were resolved. Unfortunately, Aristarchus of Samos (c 310 – c 230 BC) had proposed a heliocentric solar system 400 years early, but did not set about trying to prove it with observed data & arguments (ref [10]). Obviously, Aristarchus's view explained the inner planet motions, but the Earth, as a moving object, was a foreign concept.

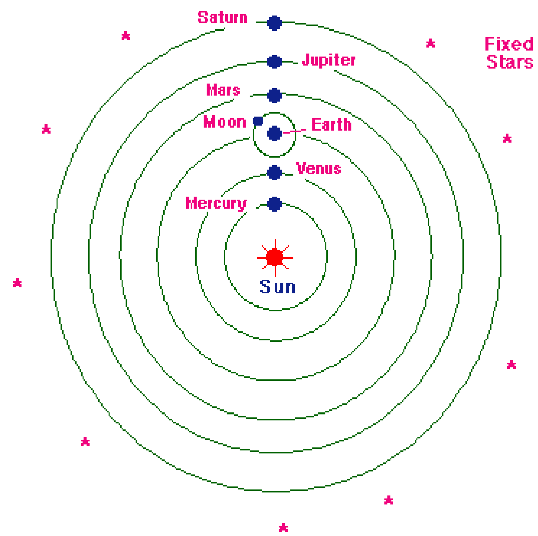
Apparent Retrograde Motion



Epicycles explained Apparent Retrograde Motion observed in some planets. Due to a bias from the primordial nebula of gas & rocks from which our solar system was formed, all planets (including Earth) orbit the Sun in a counter-clockwise fashion. The Sun & many of the solar orbiting bodies rotate around their axes in a counter-clockwise direction, also. The Sun, Moon & all visible planets cross Earth's sky from east to west in "prograde" motion along the ecliptic (ref [11]). Apparent Retrograde Motion appears in the outer planets. Mars, Jupiter & Saturn, when the Earth overtakes them in its faster orbit. Just like passing a slower car on the highway, the other car appears to go backward temporarily as our car passes. The night sky plot above illustrates a 1964 – 1965 retrograde motion of Mars (ref [12]).

Just Add Another Epicycle!

Ptolemy left 3 significant scientific treaties to history which their heirs, the Byzantines & Arabs, kept intact. As such, the documents were copied & maintained through the Roman Empire into the Middle Ages (ref [13]). More direct observations were made of planetary motion, including measuring the planets' variable speeds as they followed their actual but unknown elliptical orbits.



As the Renaissance began, a Roman Catholic canon named Nicolaus Copernicus (1473 – 1543) from Toruń, Poland was refining the latest observations to planetary motion (ref [14]). He used 48 epicycles, to match empirical celestial data of the Sun, Moon & the 5 visible planets (ref [15]). By 1514, Copernicus had begun to advance his heliocentric, or sun-centered theory in a short report that explained planetary data better than Ptolemy's geocentric system. On his deathbed in 1543, Copernicus finished his major manuscript in Latin, *On the Revolutions of the Celestial Spheres*, that proposed a heliocentric solar system & moving Earth. His solar system had a fixed Sun @ the center of the 6 known planets in their correct order orbiting the Sun in perfect circles as shown to the right (ref [16]). The stars were placed on a sphere much further from Saturn, the most distant visible planet.

Renaissance Scholars on a Moving Earth

Through the Middle Ages, monks & scholars looked to the classical antiquity of "more enlightened" Greeks & Romans as a time of knowledge & discovery relaying an entrenched dogma, whether true or not. No new ideas or understandings were anticipated. Renaissance means "rebirth" & momentum of the era allowed ***doubt*** of all passed down beliefs (ref [17]).

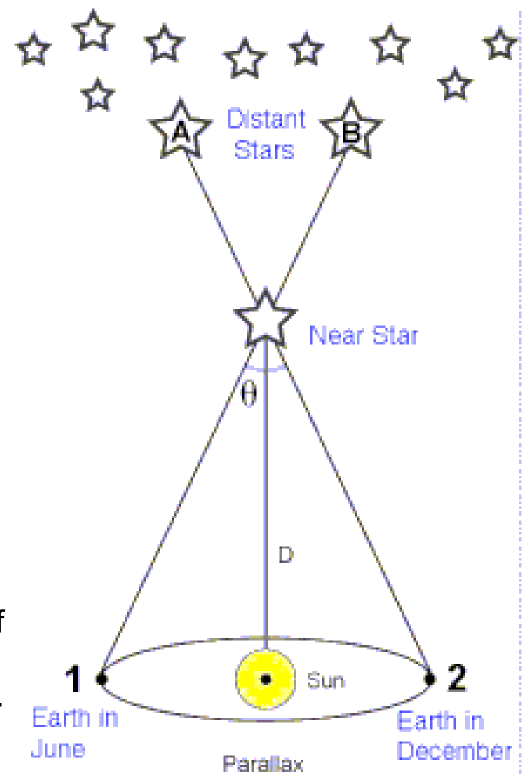
Christopher Columbus (1451 – 1506) explored the Americas in 1492 heretofore unknown to the Europeans (ref [18]). Artists, like Michelangelo (1475 – 1564) (ref [19]), emphasized realistic forms for art. Martin Luther (1483 – 1546) posted his *Ninety-five Theses* in 1517 promoting modern aspects of Protestant Christianity (ref [20]). King Henry VIII (1491 – 1547) formed the Church of England in 1532 to recognize his own divorce (ref [21]). During this period, Nicolaus Copernicus (1473 – 1543) provided a spark to doubt Ptolemy's version of our solar system.

A next generation (1546 – 1630) of astronomy contemporaries, including Tycho Brahe of Denmark, Galileo Galilei of Italy & Johannes Kepler of Germany conducted experiments, used mathematics to analyze data & test Copernicus's proposed heliocentric model.

They Did Their Best & Passed the Torch

"Science is a collection of ***good*** ideas from ***imperfect*** people."

Tycho Brahe (1546 – 1601): Tycho Brahe was born as a Danish nobleman in Knudstrup borg, Scania (now Sweden). In Danish, his name is pronounced "TEE-kow Braa" (ref [22]). Tycho promoted a pseudo geocentric-heliocentric system where the Sun orbited around the Earth & the other planets orbited the Sun. Tycho did ***not*** advance the promotion of a heliocentric solar system. However, through precise non-telescopic measurements, Tycho measured parallax of celestial objects, how these objects' locations in the sky move against the more distant, fixed background stars (ref [23]). Each object is measured in half year intervals around the Earth's orbit to maximize the parallax to observe. Along the way, Tycho measured the parallax of a supernova of 1572, that disproved the Aristotelian view of unchanging heavens. To aid in an incidental proof of a Copernican solar system, Tycho kept detailed measurements of planetary locations along their orbits (ref [24]).



BTW, in 1566 @ the age of 20, a family relative cut off part of Tycho's nose. Nerds will be nerds even 500 years ago! Tycho "drunkenly quarreled over who was the superior mathematician at an engagement party." Tycho & his 3rd cousin resolved "their feud with a duel in the dark" using swords. Tycho lost the bridge of his nose & gained a significant scar across his forehead. Using prosthetics of the day, he wore a brass nose onward (ref [25]).

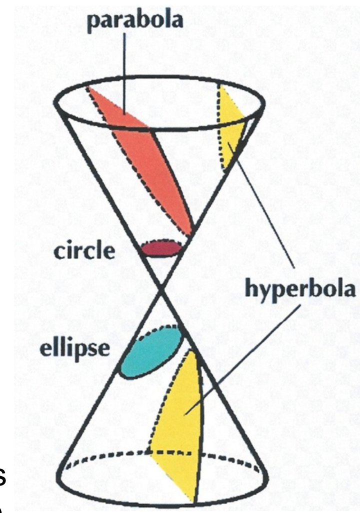
Galileo Galilei (1564 – 1642): Galileo is considered the "Father of the Scientific Method" (ref [26]). "The scientific method involves making conjectures ... deriving predictions ... carrying out ... empirical observations based on those predictions … to determine whether observations agree with or conflict with the expectations." (ref [27])

Galileo asked basic questions of motion that occur around us every day & measured / analyzed:

- an object drops accelerated in a delta-time squared (Δt^2) fashion.
- all objects drop @ about the same rate regardless of weight.
- the trajectory of a ballistic cannonball approximates an inverted parabola.
- wind resistance exists on Earth to slow moving objects to rest.
- planets may not encounter wind resistance in their orbits (ref [28]).
- a rock dropped on a moving ship has a trajectory similar to that on land (ref [29]).

Galileo heard about the invention of a telescope, built one & turned it toward the heavens (1609).

BTW, although Galileo was a pious Roman Catholic, he had two daughters & a son out of wedlock with one Marina Gamba. Due to their illegitimate births, Galileo assumed the daughters would never marry. Back then, even from a wedded couple, a daughter was an added difficulty to a family of status, requiring a large dowry to marry off properly. Since his daughters were born out of wedlock, that was another strike against them. Galileo successfully directed his daughters to join convents & to become lifetime-celibate nuns. Problem solved for dear-old-dad, Galileo! Later, Galileo's son established himself as the legitimate heir to his father's legacy (ref [30]).



Johannes Kepler (1571 – 1630): Kepler had religious convictions of a protestant Lutheran & assumed underlying mathematical laws that humans can discern with measurements. He met Tycho Brahe in 1600 (the year before Tycho's death) & saw how useful Tycho's extensive coordinate data of celestial objects were. Unfortunately, Tycho guarded his vast observations until he could reduce them mathematically to a more accurate form for publication. Much of Tycho's work was published after his death (ref [31]).

Of special interest was Tycho's data on the orbit of Mars. Apart from Venus, Mars is Earth's closest neighbor & Mars' orbit was clearly less than circular based on Tycho's observations. Renaissance astronomers correctly assumed that circular orbits & deviations from circular orbits were encompassed by "conic sections" (ref [32]). Conic sections are generated by slicing a cone @ various angles & given by the equation:

$$r(\theta) = r_0 / (1 - \varepsilon \cos\theta) \quad x = r(\theta) \cos\theta \quad y = r(\theta) \sin\theta$$

The equation of an ellipse has a constant in it termed "eccentricity" & assigned the Greek letter epsilon (ε) (ref [33]). Per the table to the right, an ($\varepsilon = 0$) indicates the orbit is a circle with radius (r_0). An ($\varepsilon > 0$) increases the ellipse stretching from a circle into a parabola @ ($\varepsilon = 1$).

The table to the right gives eccentricities from modern measurements (ref [34]) of various celestial bodies. Note, Mercury has the greatest ε-value, but it was a difficult planet for Tycho to examine. The next abnormal planet is Mars with an ε-value almost twice that of the others. Tycho's observations of Mars is the planet whose orbit data Kepler concentrated on to develop his 3 laws.

Eccentricity – ε Characteristics	
$\varepsilon = 0$	circle
$0 < \varepsilon < 1$	ellipse
$\varepsilon = 1$	parabola
$\varepsilon > 1$	hyperbola

Eccentricities of Visible Planets	
Mercury	**0.2056**
Venus	0.0068
Earth	0.0167
Moon	0.0549
Mars	**0.0934**
Jupiter	0.0484
Saturn	0.0541

Tycho died in 1601. Kepler indicated that Tycho had complained of bladder problems & died from these bladder complications. Afterwards, Kepler inherited Tycho's celestial measurements & his job in Prague as advisor to the King of Bohemia until 1612. His family moved to Linz in upper Austria where he resided until his death in 1630. In Austria, Kepler was the "imperial mathematician" for the Holy Roman (Austrian) Emperor, Matthias & taught @ the local college (ref [31]).

BTW, when you don't understand the math … gossip! Whisper, whisper, question, question: "Did Johannes Kepler kill Tycho Brahe by poison?" (ref [35]) Tycho knew Kepler for less than two years & withheld much of his data from Kepler during that time. Tycho made a singular effort to measure the data & wanted to publish the measurements himself. Tycho died at age 54. Kepler's explanation for Tycho's death seems suspicious. After Tycho's passing, Kepler inherited all of Tycho's celestial measurements, then assumed Tycho's job. However, when Tycho's corpse was exhumed in 2010, Kepler's explanation seems plausible (ref [36]). No abnormal chemical levels of poisoning were found in Tycho's beard & bone samples. We can always ask the question, but do we need to? Such gossip gives the gossipers material to elevate themselves, but they are the ones who do ***not*** make the effort to learn the math.

Kepler's 3 Laws of Planetary Motion

J Kepler
1571 – 1630

Kepler published his 1st & 2nd Laws of Planetary Motion in 1609 & his 3rd law in 1619. They are (refs [37] & [38]):

1. The orbit of a planet is an ellipse with the Sun at one focus.

2. A line joining a planet & the Sun sweeps out equal areas during equal time intervals.

3. The square of a planet's orbital period (T^2) is proportional to the cube of the length of the semi-major axis (a^3) of its orbit

Newton Invents Calculus for Copernicus

By 1620, for the sake of curiosity, a few scholars of the Renaissance had made significant contributions to determine the validity of the Heliocentric model of the Solar System:

- Were we on a moving Earth?
- Was the Sun @ the center of the Solar System?

From the 1st generation astronomers of the Renaissance, Nicolaus Copernicus (1473 – 1543) had proposed his Sun-centered theory. From the 2nd Renaissance generation of astronomers, Tycho Brahe (1546 – 1601) & Galileo Galilei (1564 – 1642) had collected empirical data to support the proposed theory, Galileo Galilei & Johannes Kepler (1571 – 1630) had developed mathematical models for the proposed theory that matched the measurements.

These scholars passed the challenge of understanding the Heliocentric Theory to the 3rd generation of Renaissance astronomers. Isaac Newton of Woolsthorpe, England (1642 – 1726) (ref [39]) had the aptitude & mindset to learn the information of the time & take up the challenge (ref [40]). He was born on Christmas Day prematurely to a father who had died 3 months earlier. At 3 years old, his mother re-married an English cleric. From 12 to 17 years of age, Newton attended formal schooling, learning Latin, Greek & Math. Newton started Trinity College @ the University of Cambridge as a "subsizar", supporting himself by performing valet duties. In 1664, Newton was awarded a 4-year scholarship for Trinity College.

Newton's time as a student was unremarkable. However, during the Great (Bubonic) Plague in London (1665 – 1666), Newton waited it out @ his home in Woolsthorpe. Here, he began to formulate Calculus & his solutions for Kepler's laws. In 1667, Newton became a fellow of Trinity College & continued to impress other academics of his work (ref [39]). In 1687, Newton published *The Mathematical Principles of Natural Philosophy* in Latin, where he described the universal Law of Gravitation & 3 Laws of Motion. In this text, he laid the foundations of Calculus & explained why planetary orbits satisfy Kepler's Laws of Planetary Motion.

The Groundwork for Newton's Proof

Vector Products Review: In Physics, a vector has 2D or 3D direction in space as well as a magnitude (ref [41]). The angle between parallel vectors is 0°. The angle between perpendicular vectors is 90°. A dot product of 2 vectors ($\boldsymbol{r}_a \bullet \boldsymbol{r}_b$) yields a scalar, whose value is the magnitude of both vectors times the cosine of the angle between them. A cross product of 2 vectors ($\boldsymbol{r}_a \times \boldsymbol{r}_b$) yields a vector perpendicular to the initial 2 vectors, whose length is the magnitude of both vectors times the sine of the angle between them (ref [42]).

$$(\boldsymbol{r}_a)^2 \equiv (\boldsymbol{r}_a \cdot \boldsymbol{r}_a) \qquad |\boldsymbol{r}_a| \equiv (\boldsymbol{r}_a \cdot \boldsymbol{r}_a)^{0.5}$$

$$(\boldsymbol{r}_a \cdot \boldsymbol{r}_a) = |\boldsymbol{r}_a||\boldsymbol{r}_b|\ cos\ \varphi_{ab} \qquad (\boldsymbol{r}_a \times \boldsymbol{r}_b) = (|\boldsymbol{r}_a||\boldsymbol{r}_b|\ sin\ \varphi_{ab})$$

$$\cos(0°) = \sin(90°) = 1$$
$$\sin(0°) = \cos(90°) = 0$$
$$\sin(180°) = \cos(270°) = 0$$
$$\cos(180°) = \sin(270°) = -1$$

Normal vectors ($\hat{\boldsymbol{n}}_r$) have unit length by definition.

$$\hat{\boldsymbol{n}}_r \equiv \boldsymbol{r} / |\boldsymbol{r}| = \boldsymbol{r} / \sqrt{\boldsymbol{r} \cdot \boldsymbol{r}} \qquad |\hat{\boldsymbol{n}}_r| \equiv (\hat{\boldsymbol{n}}_r \cdot \hat{\boldsymbol{n}}_r) = 1 \qquad (\hat{\boldsymbol{n}}_r \times \hat{\boldsymbol{n}}_r) = 0$$

$$(\hat{\boldsymbol{n}}_a \cdot \hat{\boldsymbol{n}}_b) = cos\ \varphi_{ab} \qquad (\hat{\boldsymbol{n}}_a \times \hat{\boldsymbol{n}}_b) = (sin\ \varphi_{ab})\, \hat{\boldsymbol{n}}_{ab}$$

A Constant Gravitational Force: Isaac Newton 1st developed & refined his Calculus framework on the findings of Galileo. As the "First Scientist" (ref [26]), Galileo conducted measurements answering how fast objects accelerate when dropped in the gravitational field @ the earth's surface. Galileo concluded that an inverted parabola approximates the ballistic trajectory of a cannonball. Newton was able to quite rigorously tie a constant gravitational field with a parabolic trajectory using his newly invented Calculus (ref [43]).

Gravitational Force Law: In formulating his theory of planetary motion, Newton proposed the force of "gravity" which occurs between all bodies of mass (refs [40] & [44]). For two spherical masses, the force is mutual between the masses, occurs directly along the line connecting the center of the spherical masses & drops off as an inverse square of the distance from the center of each mass. A proportional Universal Gravitational Constant (G) is included, evaluated by experiment (ref [45]).

$$\boldsymbol{F}_{Newton} = [(G\ M\ m)/(\boldsymbol{r} \cdot \boldsymbol{r})]\ \hat{\boldsymbol{n}}_r \quad \text{with} \quad \check{\boldsymbol{n}}_r \equiv \boldsymbol{r} / |\boldsymbol{r}| = \boldsymbol{r} / \sqrt{\boldsymbol{r} \cdot \boldsymbol{r}}$$

$$G = 6.67430 \times 10^{-11}\ \text{N} \cdot \text{m}^2/\text{kg}^2$$

An inverse square force argument can be alluded to from 3-Dimensional conditions (ref [46]). For a point mass, its gravitational force will be spread with a "density" over a sphere of area ($4 \pi r^2$). Then, the "density" of the force magnitude will decrease in an inverse radius squared fashion ($1/r^2$) as the sphere area & distance (r) from the point mass increases (ref [40]). Newton further assumed that for Earth's solar system, the mass of the Sun (M) is much larger than a planet's mass (m), then the Sun will essentially be stationary. The problem becomes a "central force" problem with the coordinate origin @ the Sun's center of mass.

Just as in Electromagnetism (EM) which is also an inverse square force problem, to assemble the masses (m) & (M) requires Potential Energy (PE) that can be recovered & is given by the following potential equation whose divergence gives the gravitational force vector ($\boldsymbol{F}_G$):

$$PE = -G\ M\ m / r = \Phi(\boldsymbol{r}) \quad ⇨ \quad \boldsymbol{F}_G = -[\ G\ M\ m / r^2\ ]\ \hat{\boldsymbol{n}}_r = -\nabla\Phi(\boldsymbol{r})$$

Conservation of Momentum: Conservation of linear & angular momentum (ref [47]) are expressed in the two vector equations:

$$\Sigma\ \boldsymbol{F}_i = m\boldsymbol{a} \quad \& \quad \Sigma\ \boldsymbol{\tau}_i = I\boldsymbol{\alpha} \quad \& \quad \boldsymbol{\tau}_i \equiv \boldsymbol{r}_i \times \boldsymbol{F}_i$$

$$\boldsymbol{a} \equiv \frac{d\mathbf{v}}{dt} \quad \& \quad \boldsymbol{\alpha} \equiv \frac{d\boldsymbol{\omega}}{dt}$$

In these fundamental engineering equations, all forces ($\boldsymbol{F}_i$) acting on a mass (m) determine the body's vector acceleration ($\boldsymbol{a}$), the time derivative of velocity vector ($\boldsymbol{v}$). All torques ($\boldsymbol{\tau}_i$) given by the "moment arm" ($\boldsymbol{r}_i$) & force ($\boldsymbol{F}_i$) cross product gives an angular acceleration vector ($\boldsymbol{\alpha}$) times the body's moment of inertia (I). A moment arm ($\boldsymbol{r}_i$) is the distance from the body center of mass to the point of applied force ($\boldsymbol{F}_i$). The vector ($\boldsymbol{\alpha}$) is the time derivative of the rotation vector ($\boldsymbol{\omega}$). Here, the external force of gravity ($\boldsymbol{F}_G$) is present, therefore linear momentum is ***not*** conserved. However, no external torques are present, because the central force ($\boldsymbol{F}_G$) acts on the planet's center of mass. Then, the external force moment arm is zero ($\boldsymbol{r}_G = 0$) & angular momentum ($\boldsymbol{L}$) is conserved for ***all*** central force problems.

BTW, the standard example of Conservation of Angular Momentum, is the spinning ice skater. While spinning, only gravity keeps the skater on the ice that exerts an equal but opposite force upward; virtually no torques are present. As the skater spins & draws his / her arms in, the skater's moment of inertia (I) decreases, hence an angular acceleration ($\boldsymbol{\alpha}$) increases the skater's spin ($\boldsymbol{\omega}$) to maintain a constant conserved angular momentum ($\boldsymbol{L}$).

For a central force on an orbiting body of mass (m) & velocity ($\boldsymbol{v}$),

$$\boldsymbol{L} = m\,(\boldsymbol{r} \times \boldsymbol{v}) = Constant$$

If we break the orbiting velocity ($\boldsymbol{v}$) into radial & angular components, then

$$\boldsymbol{r}(t) = r\hat{\boldsymbol{n}}_r \quad \boldsymbol{v}(t) = \dot{r}\,\hat{\boldsymbol{n}}_r + r\,\dot{\theta}\,\hat{\boldsymbol{n}}_\theta \qquad \boldsymbol{v} \equiv \frac{d\boldsymbol{r}}{dt} \quad \dot{r} \equiv \frac{dr}{dt} \quad \dot{\theta} \equiv \frac{dr}{dt}$$

$$(\hat{\boldsymbol{n}}_r \cdot \hat{\boldsymbol{n}}_r) = (\hat{\boldsymbol{n}}_\theta \cdot \hat{\boldsymbol{n}}_\theta) = 1 \qquad (\hat{\boldsymbol{n}}_r \cdot \hat{\boldsymbol{n}}_\theta) = 0$$

$$(\hat{\boldsymbol{n}}_r \times \hat{\boldsymbol{n}}_r) = (\hat{\boldsymbol{n}}_\theta \times \hat{\boldsymbol{n}}_\theta) = 0 \qquad (\hat{\boldsymbol{n}}_r \times \hat{\boldsymbol{n}}_\theta) = \hat{\boldsymbol{n}}_z$$

$$v^2 = (\boldsymbol{v} \cdot \boldsymbol{v}) = (\dot{r}\,\hat{\boldsymbol{n}}_r + r\,\dot{\theta}\,\hat{\boldsymbol{n}}_\theta) \cdot (\dot{r}\,\hat{\boldsymbol{n}}_r + r\,\dot{\theta}\,\hat{\boldsymbol{n}}_\theta)$$

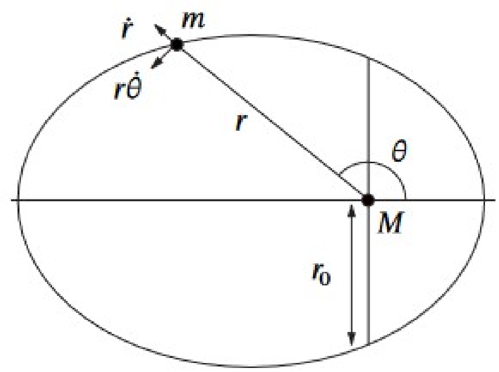
$$v^2 = \dot{r}^2\,(\hat{\boldsymbol{n}}_r \cdot \hat{\boldsymbol{n}}_r) + r^2\,\dot{\theta}^2(\hat{\boldsymbol{n}}_\theta \cdot \hat{\boldsymbol{n}}_\theta) + 2\,r\,\dot{r}\,\dot{\theta}\,(\hat{\boldsymbol{n}}_r \cdot \hat{\boldsymbol{n}}_\theta)$$

$$v^2 = \dot{r}^2 + r^2\,\dot{\theta}^2$$

$$\boldsymbol{L} = m\,[\boldsymbol{r} \times \boldsymbol{v}] = m\,[(r\hat{\boldsymbol{n}}_r) \times (\dot{r}\,\hat{\boldsymbol{n}}_r + r\,\dot{\theta}\,\hat{\boldsymbol{n}}_\theta)]$$

$$\boldsymbol{L} = m\,[r\dot{r}\,(\hat{\boldsymbol{n}}_r \times \hat{\boldsymbol{n}}_r) + r^2\dot{\theta}\,(\hat{\boldsymbol{n}}_r \times \hat{\boldsymbol{n}}_\theta)] = m\,r^2\dot{\theta}\,(\hat{\boldsymbol{n}}_r \times \hat{\boldsymbol{n}}_\theta)$$

$$L_z\,\hat{\boldsymbol{n}}_z = \mathrm{m}\,r^2\,\dot{\theta}\,\hat{\boldsymbol{n}}_z$$



In a central force problem, because the force acts between two centers of mass, the motion remains in a plane normal to angular momentum ($\boldsymbol{L}$). Only the angular component of velocity ($\boldsymbol{v}$) remains after a cross product, then:

$$L_z = m\,r^2\,\dot{\theta} \;\Rightarrow\; \dot{\theta} = L_z / (\mathrm{m}\,r^2) \quad \text{for squared speed} \quad v^2 = \dot{r}^2 + r^2\,\dot{\theta}^2$$

Conservation of Energy: Galileo made the observation that no air or wind resistance should exist to decrease a planet's motion orbiting the Sun. Then, energy of a planet should be separated between kinetic energy (KE) & potential energy (PE) (refs [40] & [48]). Furthermore, since no additional energy is added, the summation of these energies should remain constant.

$$E = KE + \mathrm{PE} \quad \text{with} \quad KE \equiv \tfrac{1}{2}\,mv^2 \;\;\&\;\; PE = -\,G\,M\,m\,/\,r$$

$$E = \tfrac{1}{2}\,mv^2 - G\,M\,m\,/\,r = \tfrac{1}{2}\,m\,[\dot{r}^2 + r^2\,\dot{\theta}^2] - G\,M\,m\,/\,r$$

Combining Conserved Angular Momentum ($L = L_z$) & Conserved Energy (E), the equation is:

$$E = \frac{m\dot{r}^2}{2} + \frac{L^2}{2mr^2} - \frac{G\,M\,m}{r}$$

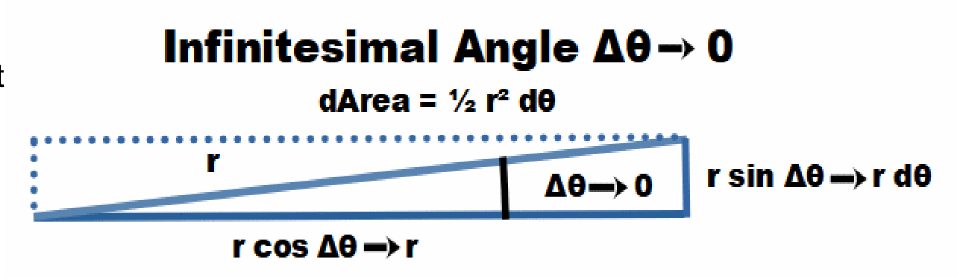
Newton's Proof of Kepler's Equations

The proof of Kepler's 3 Laws given here closely follows the article of reference [49]. Apart from algebra, a handbook of mathematical formulas (ref [41]) is useful in understanding much of the mathematics & its symbols (ref [51]). A math handbook contains integration tables, trigonometric identities & geometric descriptions of an ellipse. A complete derivation of Kepler's Laws is usually covered in a graduate level Classical Physics course with a typical text book reference [52] dedicated to the subject.

Kepler's 2nd Law: It will be shown a central force law implies a line joining an orbiting body & its attractive force origin sweeps out equal areas in equal time intervals. This law is the easiest to prove for any central force problem where angular momentum (***L***) is conserved. Noting angular velocity ($r\, d\theta/dt$) is perpendicular to the location radius (r), the conserved angular momentum length (L) for a body orbiting a central force is:

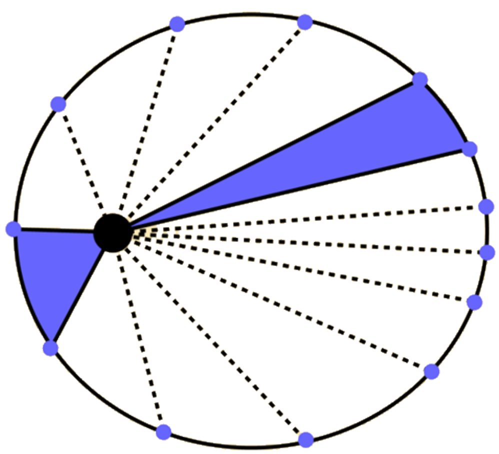
Constant = $L = m\, r^2\, \dot{\theta}$

In the limiting process, an angle ($\Delta\theta$) approaches but never reaches zero. Remember, any angle (φ) can be expressed in radians which is a ratio:



$$\phi \;=\; \Delta\theta \;=\; \frac{arc_{length}}{arc_{radius}}\;(radians) \;\Rightarrow\; \lim_{\Delta\theta\to 0}\sin(\Delta\theta) \;=\; \Delta\theta$$

The area of a right triangle is one half the area of the defined rectangle. In addition, as ($\Delta\theta$) approaches zero, the trigonometric sine of the angle approaches the arc length. The change in area is given by a limit:



$$\lim_{\Delta\theta\to 0}\frac{1}{2}r^2\sin(\Delta\theta) \;=\; \frac{1}{2}r^2 d\theta \;=\; dArea(\theta)$$

Implementing the integration angle as a function of time ($\theta = \theta(t)$) & substituting angular momentum (L = constant of motion):

$$\frac{dArea}{dt} \;=\; \frac{1}{2}r^2(t)\dot{\theta} \;=\; \frac{L}{2m} \;=\; Constant$$

The change in area swept by the orbiting body wrt time is constant, then any area between two equal delta times will be constant as illustrated (ref [53]).

$$Area \;=\; \int_{t_1}^{t_2}\frac{1}{2}r^2(t)\dot{\theta}\,dt \;=\; \frac{L}{2m}\int_{t_1}^{t_2}dt \;=\; \frac{L}{2m}(t_2-t_1)$$

Kepler's 1st Law: It will be shown an inverse-square force law implies an elliptical orbit with an inverse square force @ one focus. Combining Conserved Energy (E) & Conserved Angular Momentum (L):

$$E = \frac{1}{2}\left(m\dot{r}^2 + \frac{L^2}{mr^2}\right) - \frac{GMm}{r} \quad ⇨ \quad \dot{r}^2 = \frac{2E}{m} + \frac{2GM}{r} - \frac{L^2}{m^2 r^2}$$

$$\dot{r}^2 = \frac{L^2}{m^2}\left[\frac{2Em}{L^2} + \frac{2GMm^2}{L^2 r} - \frac{1}{r^2}\right]$$

The above equations are known as ordinary differential equations. An ordinary differential equation relates a variable & its derivatives in an equality through algebraic & many other functional forms. In addition, the dependent function derivatives are taken with respect to (wrt) a single independent variable. Here, the goal is to find a function that satisfies this non-linear 1st-order ordinary differential equation condition.

The equation is "ordinary" because time (*t*) is the only independent variable. The equation is "1st-order" because only one derivative of the variable (*r*) is required; the term "non-linear" applies because of the squared derivative & inverse function terms (ref [54]). Solving these equations are usually addressed in advanced 2nd year Calculus courses in college. The equations are sometimes called "Diff-E-Q's" by apprehensive college students. Renaming constants of motion:

$$r_0 \equiv \frac{L^2}{GMm^2} \quad ⇨ \quad GM = \frac{L^2}{m^2 r_0}$$

$$\epsilon^2 \equiv 1 + \frac{2Er_0}{GMm} \quad ⇨ \quad \frac{\epsilon^2 - 1}{r_0^2} = \frac{2Er_0}{GMm} \cdot \frac{GMm^2 r_0}{L^2} \cdot \frac{1}{r_0^2} = \frac{2Em}{L^2}$$

Substituting into the Conserved Energy (*E*) equation,

$$\dot{r}^2 = \left(\frac{L^2}{m^2}\right)\left[\frac{\epsilon^2}{r_0^2} - \frac{1}{r_0^2} + \frac{2}{rr_0} - \frac{1}{r^2}\right] = \left(\frac{L^2}{m^2 r_0^2}\right)\left[\epsilon^2 - 1 + 2\frac{r_0}{r} - \frac{r_0^2}{r^2}\right]$$

$$\dot{r}^2 = \left(\frac{L}{mr_0}\right)^2\left[\epsilon^2 - \left(\frac{r_0^2}{r^2} - 2\frac{r_0}{r} + 1\right)\right] = \left(\frac{L}{mr_0}\right)^2\left[\epsilon^2 - \left(\frac{r_0}{r} - 1\right)^2\right]$$

Making the variable change, the Conserved Energy (*E*) equation is:

$$\rho \equiv r_0/r \quad ⇨ \quad \dot{r} = \pm\left(\frac{L}{mr_0}\right)\sqrt{\epsilon^2 - (\rho - 1)^2} \quad ⇨ \quad \left[\left(\frac{L}{mr_0}\right)\left(\frac{1}{\dot{r}}\right)\right] = \pm\frac{1}{\sqrt{\epsilon^2 - (\rho - 1)^2}}$$

From Conserved Angular Momentum (*L*) equation & (ρ) substitution:

$$L = m r^2 \dot{\theta} \quad ⇨ \quad \dot{\theta} \equiv \frac{d\theta}{dt} = \frac{L}{mr^2} = \left(\frac{L}{mr_0^2}\right)\rho^2 = \left(\frac{L}{mr_0}\right)\left(\frac{\rho^2}{r_0}\right)$$

$$d\theta = \left(\frac{L}{mr_0}\right)\left(\frac{\rho^2}{r_0}\right)dt = \left(\frac{L}{mr_0}\right)\left(\frac{\rho^2}{r_0}\frac{dt}{d\rho}\right)d\rho$$

Taking the derivative of ($r(t)$) wrt ($\rho(t)$) gives:

$$\dot{r} \;\equiv\; \frac{dr}{dt} \;=\; \frac{dr}{d\rho}\frac{d\rho}{dt} \;=\; \frac{d}{d\rho}\left(r_0/\rho\right)\frac{d\rho}{dt} \;=\; -\frac{r_0}{\rho^2}\frac{d\rho}{dt} \quad \text{Inverting,} \quad -\frac{1}{\dot{r}} \;=\; \frac{\rho^2}{r_0}\frac{dt}{d\rho}$$

Substituting the time derivative of the auxiliary variable (ρ) into the Conserved Angular Momentum (L) equation:

$$d\theta \;=\; \left(\frac{L}{m r_0}\right)\left(\frac{\rho^2}{r_0}\frac{dt}{d\rho}\right)d\rho \;=\; -\left[\left(\frac{L}{m r_0}\right)\left(\frac{1}{\dot{r}}\right)\right]d\rho$$

Substituting the range time derivative ($\dot{r}$) from the Conserved Energy (E) equation:

$$d\theta \;=\; \pm\frac{1}{\sqrt{\epsilon^2 \;-\; (\rho-1)^2}}\,d\rho$$

The integral of orbit angle (θ) is solved from published integration tables (ref [41]), then using the definition of the auxiliary variable ($\rho \equiv r_0/r$) & trig identity (*cos(-φ) = cos(φ)*, ref [41]):

$$\theta \;=\; \pm\int \frac{1}{\sqrt{\epsilon^2-(\rho-1)^2}}\,d\rho \;=\; \pm\, A\cos\left(\frac{\rho-1}{\epsilon}\right) \quad ⇨ \quad \epsilon\cos\theta \;=\; \frac{r_0}{r}-1$$

$$r \;=\; \frac{r_0}{(1+\epsilon\cos\theta)}$$

For an inverse square force, an orbiting body takes on the orbit of an ellipse with the central force as one focus.

As the reader can probably note, solving a non-linear differential equation is not for the uneducated. How does one know to make the substitution ($\rho \equiv r_0/r$)? Using numerical analysis methods such as Runge-Kutta (ref [55]), most ordinary differential equations can be solved through computer programming to an approximate but specified accuracy. Numerical Analyses were developed on early IBM 360 “mainframes” to aid in orbital mechanics calculations of spaceflight (ref [56]). To get an answer of complicated ordinary differential equations in closed form with ***all*** solutions, one may need to ask a mathematician. Once a mathematician supplies the answer, the solution can always be substituted in the differential equation for a check. Here,

$$\dot{r} \;\equiv\; \frac{dr}{dt} \;=\; -\frac{r_0\epsilon\sin\theta\,\dot{\theta}}{(1+\epsilon\cos\theta)^2} \;=\; -\left(\frac{r^2\dot{\theta}}{r_0}\right)\epsilon\sin\theta$$

From conservation of angular momentum (L) cited previously:

$$\dot{\theta} = L/m r^2 \quad ⇨ \quad \dot{r} \;=\; -\left(\frac{L}{m r_0}\right)\epsilon\sin\theta \;=\; \pm\left(\frac{L}{m r_0}\right)\epsilon\sqrt{1-\cos^2\theta}$$

which when substituted satisfies the orbital equation cited previously:

$$\dot{r}^2 = \left(\frac{L}{m r_0}\right)^2 \left[\epsilon^2 - \left(1 - \frac{r_0}{r}\right)^2\right]$$

We should take a moment to ***appreciate*** the mathematical gymnastics Isaac Newton had to overcome to solve this very difficult differential equation in Calculus. In comparison to his historic equal in Physics, i.e., Albert Einstein, the mathematics for Einstein's Special Relativity is no more difficult than a square root. The math is complicated for General Relativity, but Einstein borrowed existing mathematical research for that. Not only did Newton invent an entire branch of mathematics, but applied the math in a problem now considered of elevated difficulty in a field he just invented. Thank you Isaac Newton!

Runge-Kutta Methods (pronounced RUUNG-a KUUT-tah) use numerical computation techniques to approximate a solution to an ordinary differential equation based on known conditions @ one independent point. These techniques are limited to arithmetic operations & let computer calculations approximate an equation solution @ another independent point. The technique predates the computer age & was developed by two Germans, Carl Runge (1856 – 1927) & Wilhelm Kutta (1867 – 1944), around 1900 (ref [50]).

Kepler's 3rd Law: It will be shown an inverse square force law implies that the square of the period (T^2) of an orbit is proportional to cube of the length (a^3) of the semi major axis. From the Conservation of Angular Momentum & the proof of Kepler's 2nd Law, the rate of change of area swept by an angle wrt time is:

$$\frac{dArea}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{L}{2m} = Constant$$

The area of an ellipse with semi major (a) & minor (b) axes is (ref [41]):

$$Area = \pi ab = \pi a^2 \sqrt{1-\varepsilon^2} \quad ⇨ \quad \frac{Area}{Time} = \frac{L}{2m} = \frac{\pi a^2 \sqrt{1-\epsilon^2}}{T}$$

$$T = 2\pi a^2 \sqrt{1-\epsilon^2} \left(\frac{m}{L}\right) \quad ⇨ \quad T^2 = 4\pi^2 a^4 (1-\epsilon^2) \left(\frac{m}{L}\right)^2$$

Expressing angular momentum constant (L) as length in terms of semi-major axis (a) through the geometric definition of (r_0):

$$r_0 \equiv \frac{1}{GM} \left(\frac{L}{m}\right)^2 = a(1-\epsilon^2) \quad ⇨ \quad \frac{1}{GM} = a(1-\epsilon^2) \left(\frac{m}{L}\right)^2$$

Combining the above Ellipse Area & Angular Momentum Conservation equations:

$$T^2 = \left(\frac{4\pi^2}{GM}\right) a^3$$

As Newton promised, the orbital period squared (T^2) is proportional to the elliptical orbit's semi-major axis cubed (a^3). Newton even provided the constant of proportionality which is dependent on the Sun's mass (M) & other universal constants.

Conclusion

Before Isaac Newton, things happened because we were @ the whim of the gods who acted in the moment. After Newton, but before Planck, we lived in too much of a pre-determined machine. Much of our trajectories were decided upon @ the universal creation before we were even born. With Bell's Theorem & "no hidden variables", we may have a choice again! (ref [57]) In 1926, when he saw Physics may incorporate randomness, Albert Einstein (1879 – 1955) who earned his doctoral degree in a totally Newtonian world remarked, "He [God] does not play dice with the universe." (ref [58])

This article covers a "thread of knowledge" handed down through three generations across Europe during the Renaissance. The people who added to the thread had childhood educations & were "where the knowledge was". More importantly, they sacrificed time & effort of themselves to learn the present technical knowledge of the day. We should not call them heroes; nor do I think they want to be remembered as such. They ***did not*** risk & sacrifice their lives so that others could live.

The early scientists covered in this article ***did*** want to add to the technical knowledge of human kind & "leave the world ***smarter*** than they found it." In stumbling on the mechanics of this universal machine we inhabit, they made the world a ***better*** place. By setting out to prove a hypothesis, taking empirical data, determining rules that govern the data, then inventing a math framework to understand the rules, their efforts have allowed for many of the engineering advancements since the 18^{th} century. Modern Science / Technology / Engineering / Mathematics (STEM) majors should ***aspire*** to contribute meaningful accomplishments to posterity as these primordial STEM scholars did.

Joining a temporal team across generations, Newton added ***greatly*** to the knowledge of human civilization. In 1675, a humble Isaac Newton wrote his famous quote: "If I have seen further it is by standing on the shoulders of Giants." (ref [59]) Now, you know the names of some of the scientific giants on whose shoulders Isaac Newton stood.

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