

Quanta Elucidates Spectral Radiation

Introduction

Presently, there are two separate worlds in Modern Physics:

“Ping Pong Ball” World: There is the “Ping Pong Ball” World of Classical Physics in which we live (refs [1] & [2]). In this macroscopic world, one assumes that many processes can be broken down into very small indestructible “ping pong ball” like particles that interact, bounce around, attract or repel each other in solids, liquids & gases. Then, derived approximations can be successfully applied. Indeed, [Isaac Newton](#) (1642 – 1726) (ref [3]) published *Mathematical Principles of Natural Philosophy* in 1687 (ref [4]) describing the Classical Mechanics of a “Ping Pong Ball” World almost in total which lasted over two centuries. However, on nanometer ($1 \text{ nm} = 10^{-9}$ meters) atomic scales & below, it’s a crude approximation (ref [5]).

Classical fluid flow & its continuous liquid assumptions, can be seen as an estimation when the diameter of ping pong balls approaches zero (ref [6]). Likewise, major concepts of Special & General Relativity from [Albert Einstein](#) (1879 – 1955) (ref [7]) are based in this world (refs [8] & [9]). Some preconceived notions of reality have to be modified for Einstein’s theories, but other more concrete reality assumptions hold fast.

Quantum World: There is the “Quantum” World that is totally driven by interpretation of experimental data @ the atomic scale (refs [10] & [11]). If a Physics Professor tells you some unbelievable aspect of the Quantum World, when he / she is smiling, it may be a prank. With a “straight face”, it’s the latest outlandish deduction from current experimental results. The following quotes are from Nobel Laureates in the field of Quantum Theory (ref [12]):

“If you can think about Quantum Theory without getting dizzy, you don’t get it!”
Niels Bohr (1885 – 1962) (ref [13])
Physics Nobel Laureate of 1922.

“I think I can safely say that nobody understands Quantum Mechanics!”
Richard Feynman (1918 –1988) (ref [14])
Physics Nobel Laureate of 1965.

Directly from the Nobel Laureates, Quantum Theory seems incomplete! A Physics Professor reported to me one college student who waded through the logic of [Classical Mechanics](#) & [Classical Electromagnetism](#) (EM), then received a great shock from the 1st weeks of Quantum Mechanics (QM) & abruptly changed majors. With the acceptance that electrons only rotate @ one “quantized” speed, or the correlation of the Periodic Table & “quantized” electronic orbits around the atom, one can see QM is inexplicably “spot on” among its many applications (ref [15]).

No Hidden Variables to the Rescue: If a Physics student is waiting for “hidden variables” to be identified that merge QM into the logic & elegance of Classical Physics

(ref [16]), it's a false hope! I would have benefited from a preview of this atomic reality in high school. Give the grade school Science teachers an overall understanding of QM.

For example, when the high school Science teacher presents the Periodic Table, some spherical harmonic orbits of electrons should be illustrated to show the pupil, this is how an electron “orbits” its atomic nucleus (ref [17]). The Science teacher can simply “explain” that the orbits are in excellent agreement with experimental data. More importantly, this morsel of subatomic reality may spark a number of curious students to delve further into the information of atomic electron orbits & put those students on a lifetime of pursuit of scientific knowledge ... a Scientist / Technology / Engineering / Mathematics (STEM) career in the making!

Divisor of the Worlds: The “Planck” Constant is named after the German Physicist Max Planck (refs [18] & [19]), considered the “Father of Quantum Theory.” (ref [20]) Herr Planck was German & the German “a” in “Planck” has an “ah” sound as in “water” (refs [21] & [22]).

Anytime the Planck Constant (h) appears in an equation, concepts of Quantum Theory are applied (ref [23]). If this constant is absent, usually, the assumptions of Classical Physics are valid. The Planck Constant has a minuscule value & units ([si4x6.pdf](#)):

$$h = 6.62607015 \times 10^{-34} \text{ (J/Hz)}$$

BTW, Hitler embraced QM & shunned Einstein’s Relativity theories because Einstein was Jewish (ref [24]). However, Max Planck raised his family to view Hitler’s Germany as a Germany to resist. In fact, one of Max Planck’s son, Erwin, was executed on 23 January 1945 because of his involvement in a failed assassination attempt of Hitler (ref [25]). Science always attempts progress, **despite** struggles against reactionary events on the World Stage.

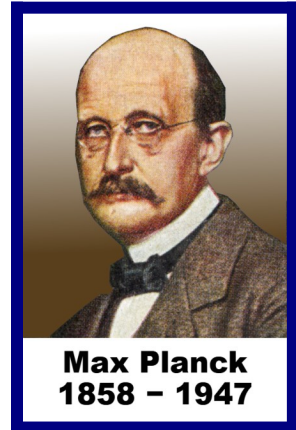
“Ping Pong Ball” 4-Vectors: In Einstein’s [Special Relativity](#) (SR), the overriding characteristic constant is (c), the speed of light in a vacuum ([si4x6.pdf](#)):

$$c = 299792458 \text{ (m/s)}$$

In many SR equations, if velocity appears, it’s a fraction of light speed. The magnitudes of a couple of descriptive 4-vectors are conserved in inertial reference transformations. Magnitudes of 4-vectors, time-location (ct , \mathbf{x}) & energy-momentum (E/c , \mathbf{p}), are conserved in SR (ref [26]).

Fuzzy Quantum World: In the Quantum World, the Heisenberg Uncertainty Principle applies. This principle was proposed by Werner Heisenberg (1901 – 1976) (refs [27] & [28]) in 1927. An interplay exists between a quantum particle’s similar descriptive 3-vector momentum (\mathbf{p}) & location (\mathbf{x}) or the particle’s energy state (E) & time (t) of the measurement (refs [29] & [30]).

$$\Delta x_i \Delta p_i \geq h/(4\pi) \quad \& \quad \Delta t \Delta E \geq h/(4\pi)$$



These variable duos were termed “complimentary pairs” by Neils Bohr (ref [31]). If a quantum particle’s location is known with great precision, where the particle is going is not known. If a particle has a measured energy state (E), it applies somewhere in a long time interval (Δt). Furthermore, these uncertainties are due to the wave nature of subatomic particles, not the measurement’s change of a particle’s physical state (ref [32]).

QM incorporated the 4-vectors of classical SR into its quantum wave equations a couple of decades after Planck proposed quantum states in photons (ref [33]). The results were ever more “bizarre pandemonium” ... particles with negative kinetic energy deemed antiparticles made of antimatter. For QM, the more outlandish the claim, the more probable it’s predictions! Characteristics of these common antimatter particles have been verified in high-energy atomic colliders. In fact, these antiparticles are produced frequently by high-energy cosmic rays (ref [23]) raining on us in nature. They quickly disappear in an almost undetectable flash when encountering their diametric matter counterparts.

Two Models: To agree with QM experimental results, the QM Physicist uses one of two models, the wave model or the particle model. Certain rules exist as to when to apply each model. The “Ping Pong Ball” model is similar to the particle model except the QM particles are indistinguishable. One can mark a “ping pong ball” & track it. In the QM World, that’s not possible! The wave model is driven by experiment & uses equations that apply in wave optics & waves on the surface of water (ref [34]).

Considering the type of problem to be solved, the QM Physicist “assumes” a QM atomic particle is described as either a “wave packet” or “indistinguishable particle”, applies the corresponding model & correctly predicts a measurable result. However, the consensus from advanced Physics is, an electron “is like neither” wave nor particle (ref [35]). “In reality,” the double-slit experiment shows this duality & “contains the *only* mystery” of QM (ref [36]). Decoherence (ref [37]) examines this issue, further.

Early Photon Experiments

When Physicists tried to explain the spectral behavior within the visible spectrum of the Sun & other heated objects, they were dabbling in spectral frequencies & associated wavelengths in nanometers ($1 \text{ nm} = 10^{-9} \text{ m}$). Eventually, they would stumble on to the Quantum Rules Mother Nature applies to the atomic scale of the Universal Machine we inhabit.

Wave Description: A wave moving through space with velocity (c) has an Amplitude (A) & wavelength (λ) or frequency (ν) given by the relationship ($c = \lambda\nu$). This equation is readily shown when peak-to-peak amplitude is wavelength & moves through space @ (Δt), then ($\lambda = c\Delta t$) or ($c = \lambda/\Delta t$) or ($c = \lambda\nu$). Frequency ($\nu = 1/\Delta t$) is expressed in Hertz (Hz = cycles/second).

Newton's Prism:

Between his periods of formulating Calculus & Celestial Mechanics, Isaac Newton lectured on prisms & the nature of light (ref [23]). Around 1670, Newton formulated his Theory of Color. Newton used a glass prism to spread white light into its many spectral colors, then reflected, scattered & transmitted through another prism, isolated

colored light beams to determine that spectral light retains its color. Newton argued that discrete corpuscles or particles of light decelerated @ different rates in a prism, thus causing a spectrum.

Newton's Conclusion: The color of spectral light is a property of discrete light corpuscles, not the prism.

Light has some bending or "refraction" in a glass prism. This angle of refraction was 1st described by the Persian mathematician Ibn Isahl (c 940 – c 1000). The law has been rediscovered several times, but now assumes the Renaissance name Snell's Law from Willebrord Snellius (1580 – 1626) (ref [38]). For the refraction schematic to the right, ($v_1 > v_2$) light travels faster through the *white* medium #1 into the *blue* medium #2.

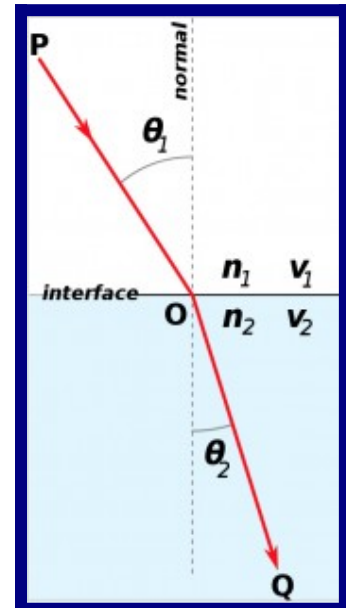
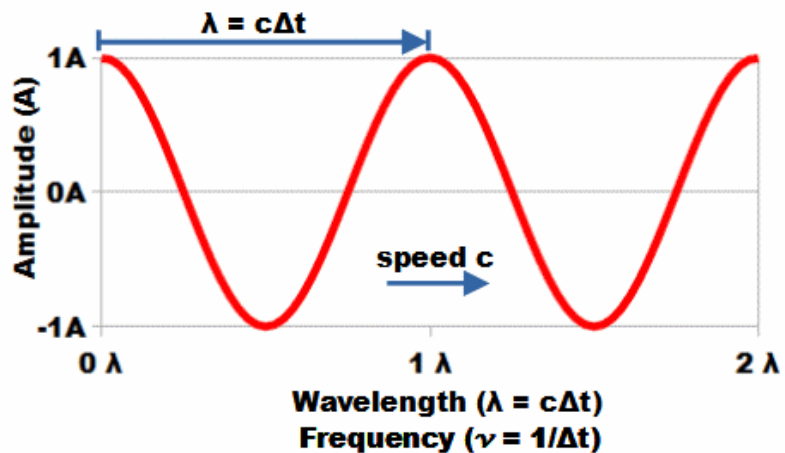
$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{n_1}{n_2} = \frac{c/v_1}{c/v_2} = \frac{v_2}{v_1}$$

The index of refraction (n_i) is expressed as ($n_i = c/v_i$), the speed of light in a vacuum (c) divided by a slower speed of light in a transparent medium (v_i). Here, (n_i) is always greater than or equal to unity ($n_i \geq 1$). From observed Sun light refraction ($n_{blue} > n_{red}$), **blue** light has a greater refraction ($\theta_{blue} > \theta_{red}$) & travels slower in a prism than **red** light ($v_{blue} < v_{red}$).

Newton's Luck: Newton was right about **blue** light traversing @ slower speeds than **red** light in a prism. However, as we shall see, Max Planck had to assume **blue** photons are more energetic than **red** photons & travel with greater momentum. If anything, the **blue** photons should traverse the prism with less time & refraction.

That's just **not** the case! Maxwell's Equations dictate that light travels through a vacuum

Wave Motion in Space

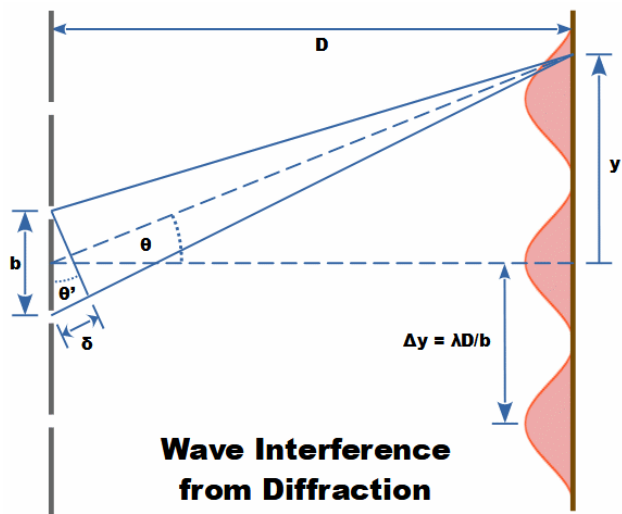


via photons @ constant speed (c) via electrical & magnetic fields. “Waves of electricity ... create waves of magnetism, which go on to make waves of electricity.” (ref [39]) To a photon is ascribed a frequency & wavelength as evaluated by a trigonometric sine wave. A prism divides white light into spectral colors. In the prism silicate structure, each spectral light beam photon collides with outer orbiting electrons of individual glass silicon & oxygen atoms. The blocking atoms individually absorb all photons traversing the prism. The energized electrons remain in an elevated state for a brief specific time, then re-emit the photon in its original direction settling down to a pre-absorption stable state (refs [23] & [40]).

Very little heat is generated in the glass because of the light beam’s traversal. The key is that photons with higher frequencies energize orbiting electrons more & the time delay for their re-transmission is greater. **Blue** photons have shorter wavelengths / greater energies, travel “slower” through glass & are refracted more. **Red** photons have longer wavelengths / less energies, travel “faster” through glass & are refracted less.

Among other things, Newton was a **lucky** guy! However, Isaac Newton had his own “epicycles”; check out Newton’s alchemy beliefs (ref [41]). Those beliefs, common of the day, show the fallibility of a guy, not so different from you or me!

Fraunhofer’s Diffraction Grating: Joseph von Fraunhofer (1787 – 1826) (ref [42]) made the next great advance in spectroscopy in perfecting diffraction gratings (ref [43]). Thomas Young (1773 – 1829) (ref [44]) disregarded Newton’s corpuscular light theory & assuming a luminous wave description, proved the spectral pattern given by a diffraction grating in 1799. By 1814, Fraunhofer refined the spectroscopy using diffraction gratings. The spectroscopy became a laboratory instrument able to analyze a reproducible spectrum for analyses.



Fraunhofer’s Conclusion: The color of spectral light is a property of the light’s individual wavelengths which combine to form white light.

$$\tan \theta = y/D \quad \& \quad \sin \theta' = \delta/b \quad \& \quad \delta_{max} = m\lambda$$

$$D \gg y \quad \& \quad \tan \theta \equiv \sin \theta / \cos \theta \quad \hookrightarrow \quad \sin \theta' \quad \hookrightarrow \quad \theta$$

$$\theta \approx y/D \approx \delta/b = m\lambda/b \quad \hookrightarrow \quad m\lambda/b \approx y/D \quad \hookrightarrow \quad y = m\lambda D/b = mcD/\nu b$$




The above calculations reflect the typical 2 slit interference illustration (refs [45] & [46]) to the right. Many narrow parallel slits can be machined into a metal plate to produce the grating. The width of the slits are uniform & significantly smaller than the uniform separation distance between the slits. Spectral refraction is linear due to frequency &

not due to non-linear effects as in a glass prism.

Spectral Light Sources & Intensities & Defects

Once the task of producing visible spectra was solved well enough to satisfy repeatability for laboratory settings, examination of spectra from different light sources & spectral intensities & spectral defects began in earnest.

Gustav Kirchhoff: As Thermodynamics & Spectroscopy progressed through the 19th century, Gustav Kirchhoff (1824 – 1887) (ref [47]) investigated spectral emissions of many substances, discovered a few unknown elements & developed his 3 Laws of Spectroscopy:

Kirchhoff's 3 Laws of Spectroscopy	
<p>1) An incandescent solid, liquid or gas under high pressure emits a continuous spectrum.</p>	<p>Continuous spectrum</p> 
<p>2) A continuous spectrum source viewed through a cool, low-density gas produces an absorption-line spectrum.</p>	<p>Absorption spectrum</p> 
<p>3) A hot gas under low pressure emits a "bright-line" or emission-line spectrum.</p>	<p>Emission spectrum</p> 

In these 3 laws are the beginning of modern Spectroscopy, identifying any substance & its composition based on its emitted spectra (ref [48]). Kirchhoff also applied Thermodynamics to spectral emissions. He refined the definitions of spectral absorption (α_ν), emission (ϵ_ν), reflectance (ρ_ν) of heated objects undergoing incandescence:

$$\alpha_\nu \equiv \frac{E_{e,\nu,absorb}}{E_{e,\nu,incident}} = 1 - \rho_\nu \quad \& \quad \rho_\nu \equiv \frac{E_{e,\nu,reflect}}{E_{e,\nu,incident}} = 1 - \alpha_\nu \quad \& \quad \epsilon_\nu = \frac{M_{e,\nu,emit}}{M_{e,\nu,ideal}}$$

$$E_{e,\nu,incident} = E_{e,\nu,absorb} + E_{e,\nu,reflect}$$

Incandescent objects have a noticeable “glow” or light emission due to their elevated temperatures. Above, spectral exitance ($M_{e,\nu}$) is the spectral radiant energy emitted per unit area per unit time @ a given frequency (ν) (ref [49]). Spectral irradiance ($E_{e,\nu}$) is the spectral radiant energy incident on an object per unit area per unit time @ a given frequency (ν). As the numerical behavior of Kirchhoff cavity radiation was quantified, it was this spectral exitance “glow” that Max Planck was trying to model from 1st principles, that required assumptions of QM (ref [23]).

Preliminary Behavior of Kirchhoff Radiation

Absolute Temperature: Kirchhoff radiation of substances is identified by thermodynamic states. The fundamental descriptor for these states is absolute

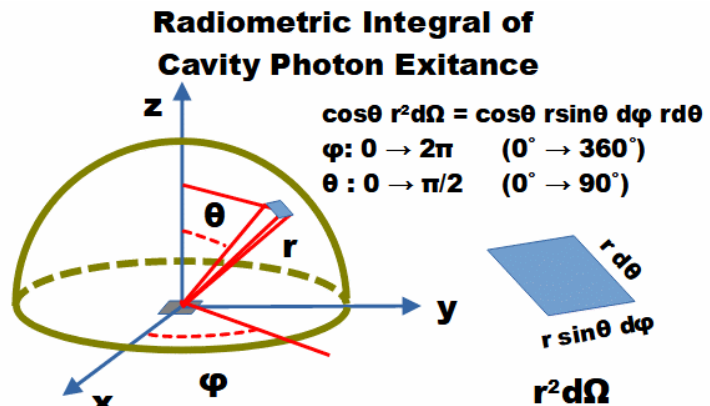
temperature as specified by the 3rd Law of Thermodynamics (ref [50]). Whereas Celsius & Fahrenheit have arbitrary origin zero's for convenience & can be negative, absolute thermodynamic temperatures (Kelvin & Rankine) are always positive with **no** degree symbol (°) (ref [51]).

In a thermodynamic temperature equilibrium with surrounding objects, the object's spectral absorption divided by its spectral emission will converge to unity. As the absorption value alone approaches unity, the object's surface takes on characteristics of a Kirchhoff radiator. Then, any incident radiation falling on a Kirchhoff radiator's surface will be completely absorbed & re-emitted as radiant energy based on the Kirchhoff radiator's thermodynamic temperature. The ideal of a Kirchhoff radiator is approached but never completely achieved in nature. Absorption (α_ν) & emissivity (ϵ_ν) of around 95% is measured for typical near Kirchhoff radiators (ref [52]).

Kirchhoff Cavity Radiator: To best describe Kirchhoff radiation, a cavity radiator was devised whereby a cavity of macroscopic size is created in a uniform material & allowed to obtain thermodynamic equilibrium. The cavity also has a small hole whereby small amounts of equilibrium radiation can be emitted & measured (ref [53]). Due to these requirements @ the cavity hole, incident radiation enters, is absorbed by the cavity & surrounding material, but no part of that incident radiation is reflected. In this ideal Kirchhoff radiator, an ideal 100% absorption (α_ν) has been achieved @ the cavity hole. The cavity has a specific radiation density within as radiation bounces back & forth between the cavity walls.

Kirchhoff Cavity Spectral

Exitance: The Kirchhoff spectral exitance ($M_{e,\nu}(T)$) emitted from the spectral radiant energy density ($w_{e,\nu}(T)$) in the cavity is described by a characteristic absolute temperature (T). The exitance is a radiant power loss from the cavity or its spectral density times the exit velocity (c) times a reduction factor of the cavity. The exit velocity (c) is the speed of EM radiation in the medium.



A uniform radiant energy density ($w_{e,\nu}(T)$) of photons is assumed in the cavity, then the illumination evaluation of scene rendering (ref [54]) can be used to determine a form factor. Using integral Calculus, incident photons are summed over a hemisphere of radius (r). The integrating solid angle is taken as ($r^2 d\Omega = r^2 \sin\theta d\phi d\theta$) with (θ) as the polar angle, (ϕ) as the azimuth angle. The incident unit area decreases with polar angle as ($\cos\theta$). For a uniform photon density, the hemisphere integral is ($(\pi r^2)/(4 \pi r^2) = 1/4$). The denominator of this form factor is a normalizing factor, the entire sphere area ($A_{sphere} = 4 \pi r^2$) to consider all photon orientations. Then,

$$M_{e,\lambda}(T) = (c/4) w_{e,\lambda}(T) \quad \& \quad M_{e,\nu}(T) = (c/4) w_{e,\nu}(T) \quad (\text{W/m}^2)$$

To match Solar spectral irradiance ($E_{e,\lambda}$) as measured @ the Earth, Solar spectral exitance ($M_{e,\lambda}$) is multiplied by the Sun's radius (r_{Sun}) : Earth's mean orbit (R_{Earth}) ratio squared (ref [55]).

$$\rho \equiv (r_{Sun} / R_{Earth})$$

$$E_{e,\lambda}(T) = \rho^2 M_{e,\lambda}(T) d\lambda \quad \& \quad E_{e,\nu}(T) = \rho^2 M_{e,\nu}(T) d\nu \quad (W/m^2)$$

To convert from spectral wavelengths (λ) to spectral frequencies (ν), or vice versa, using differential Calculus:

$$c = \lambda \nu \quad (m/s)$$

$$\nu = c / \lambda \quad \Leftrightarrow \quad \frac{d\nu}{d\lambda} = \frac{d}{d\lambda}(c/\lambda) = -\frac{c}{\lambda^2} \quad \Leftrightarrow \quad d\nu = \frac{d\nu}{d\lambda} d\lambda = -\frac{c}{\lambda^2} d\lambda$$

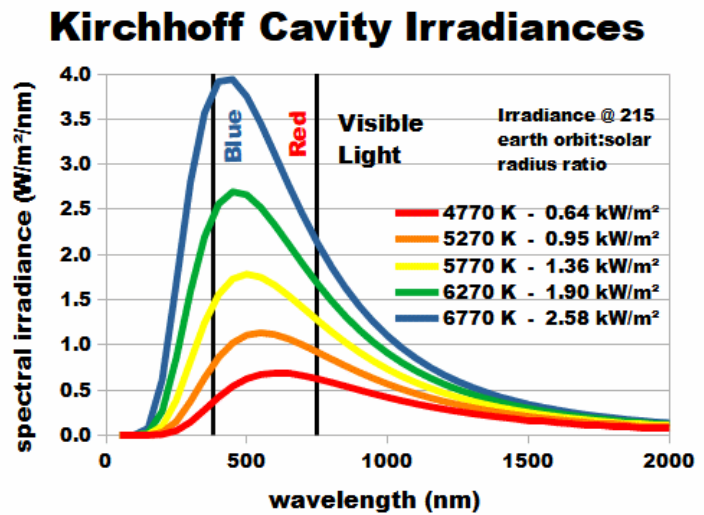
$$\lambda = c / \nu \quad \Leftrightarrow \quad \frac{d\lambda}{d\nu} = \frac{d}{d\nu}(c/\nu) = -\frac{c}{\nu^2} \quad \Leftrightarrow \quad d\lambda = \frac{d\lambda}{d\nu} d\nu = -\frac{c}{\nu^2} d\nu$$

Empirical Data Fit: The spectral intensity data sets were known for Kirchhoff radiation @ various absolute temperatures (T). Max Planck developed an empirical curve-fit of these data sets before he proved his coefficients theoretically. The negative sign in the frequency \leftrightarrow wavelength conversion above indicates that integration limits of spectral functions have an inverse relationship with frequency & wavelength ($c = \nu \lambda$). Because integration limits are from 0 to ∞ , the minus sign is ignored. The following are constants below: B_ν & β_ν For spectral energy densities ($w_{e,\nu}(T)$) & ($w_{e,\lambda}(T)$) within the Kirchhoff cavity:

$$w_{e,\nu}(T) = B_\nu \nu^3 / [\exp(\beta_\nu \nu / T) - 1] \quad \& \quad w_{e,\lambda}(T) = B_\nu (c/\lambda)^5 / [\exp(\beta_\nu c/\lambda T) - 1]$$

Exact numerical measurements of spectral EM radiation outside the visible range were limited in Planck's QM data sets. However, the data sets were complete enough to verify the above function form.

BTW, our visible light range as shown on the above plot @ 5770 K is between 380 nm (blue) to 750 nm (red) in wavelength (ref [56]). Our Sun's (effective) surface temperature which approximates a Kirchhoff cavity radiator is measured @ about 5772 K (ref [57]). From the above plot, our Darwinian sight locates the Sun's maximum spectral exitance ($M_{e,\nu}$) @ about the midpoint of our visible window.



Visible & IR Windows					
color	range	wavelength (μm)	(nm)	frequency (THz)	Temperature (K) (°F)/(°C)
blue	lo	0.38	380	789	
yellow	mid	0.58	580	517	5700 K 9800° F
red	hi	0.75	750	789	5400° C
near	lo	3.0	3000	100	770 K 930° F
infrared	hi	5.0	5000	60	500° C
far	lo	8.0	8000	38	280 K 50° F
infrared	hi	14.0	14000	21	10° C

Wien's Displacement Law: The law states the location of spectral maximum exitance ($M_{e,\nu}$) of Kirchhoff cavity radiation is proportional to its inverse absolute Temperature (T) (ref [58]). In 1893, Wilhelm Wien (1864 – 1928) derived the law using Thermodynamic arguments. “The adiabatic principle allowed Wien to conclude that for each mode, the adiabatic invariant energy / frequency is only a function of the other adiabatic invariant, the frequency / Temperature.”

A standard exercise of Calculus is to take the derivative of a function & set the derivative equal to zero. If the derivative of Kirchhoff spectral exitance ($M_{e,\nu}$) or ($M_{e,\lambda}$) is taken, set to zero, then solved for frequency or wavelength, that result is where the spectral Kirchhoff cavity exitance ($M_{e,\nu}$) or ($M_{e,\lambda}$) is @ a maximum. A web calculator is on the internet (refs [59] & [60]).

$$d/d\nu [M_{e,\nu}(T)] = 0 \quad \Leftrightarrow \quad \nu_{max} = C_\nu \cdot (T) \quad \Leftrightarrow \quad \nu_{max} = [0.058789 \text{ (THz/K)}] \cdot T \quad (\text{THz})$$

$$d/d\lambda [M_{e,\lambda}(T)] = 0 \quad \Leftrightarrow \quad \lambda_{max} = C_\lambda / (T) \quad \Leftrightarrow \quad \lambda_{max} = [2897800.0 \text{ (K}\cdot\text{nm)}] / T \quad (\text{nm})$$

The maximum spectral values differ somewhat between choice of Kirchhoff spectral exitance functions to analyze. Our Sun’s surface temperature which approximates a Kirchhoff’s cavity radiator is measured @ about 5772 K (ref [57]). When the Kirchhoff spectral wavelength exitance ($M_{e,\lambda}(T)$) is analyzed for its spectral maximum, $\lambda_{max} = 502$ nm, somewhere between **orange** & **yellow** on our visual color spectrum (ref [56]).

Stefan – Boltzmann Law: The law states total Kirchhoff exitance varies as (T^4) (ref [61]). In 1864, John Tyndall (1820 – 1893) “presented measurements of the **infrared** emission by a platinum filament and the corresponding color of the filament.” From this data, Kirchhoff cavity exitance ($M_e(T)$) as a function of “the fourth power of the absolute temperature was deduced by Josef Stefan (1835 – 1893) in 1877.”

$$\int M_{e,\lambda}(T) d\lambda = \int M_{e,\nu}(T) d\nu \equiv M_e(T) = \sigma T^4 \text{ (W/m}^2\text{)} \quad \text{with} \quad [\lambda: 0 \rightarrow \infty \quad \& \quad \nu: 0 \rightarrow \infty]$$

$$E_e(T) = \rho^2 M_e(T) = \rho^2 \sigma T^4 \text{ (W/m}^2\text{)} \quad \text{with} \quad \rho \equiv (r_{Sun} / R_{Earth})$$

The above spectral irradiance plot lists fourth power temperature irradiance ($E_e(T)$) using the same Sun’s radius (r_{Sun}): Earth’s mean orbit (R_{Earth}) ratio in the Stefan – Boltzmann Law for the listed absolute temperatures.

The Equipartition Distribution – A Classical False Start

In 1900, Max Planck published his formulation of Kirchhoff cavity exitance ($M_e(T)$) that assumed discrete or quantized spectral energy densities ($w_{e,\nu}$) within an ideal Kirchhoff cavity radiator. The proof given here follows ref [62]. Planck initially assumed Kirchhoff radiation density ($w_{e,\nu}$) within the cavity would be composed of photons with discrete

frequencies (ν_m) given by:

$$\nu_m = [2c / \lambda_{max}] m \quad m = 1, 2, 3, \dots$$

The maximum wavelength (λ_{max}) is a characteristic of a typical cavity dimension (L_{cavity}). This requirement establishes a lower limit for radiation frequency. As we shall see, arguments from statistical mechanics show that an infinite amount of energy could be stored in Kirchhoff radiation within the cavity as photon wavelengths are subdivided, smaller without limit.

Equipartition Theorem: This approach counts the modes or degrees of freedom within a uniform physical substance & divides the total energy of the substance equally among these degrees of freedom (ref [63]).

Ideal Gas Example: In an [ideal gas](#), where the molecules have only translation velocity (\mathbf{v}) as an energy component (no rotation), the internal Kinetic Energy (KE) of each molecule is:

$$KE_{molecule} = \frac{1}{2}m_g v^2 = \frac{1}{2}m_g[(v_x)^2 + (v_y)^2 + (v_z)^2] = (3/2) m_g \langle v^2 \rangle_{avg} \quad (\text{J})$$

Here, (v_i) is a vector component of the molecule 3D velocity (\mathbf{v}) with a velocity average (v_{avg}), (m_g) is the mass of each molecule. From experiment ([si4x6.pdf](#)), the Boltzmann Constant (k_B) relates the [absolute Temperature](#) (T) of the ideal gas & molecular KE, then (ref [64]):

$$KE_{molecule} = (3/2) m_g \langle v^2 \rangle_{avg} = (3/2) k_B T \quad \text{with} \quad k_B = 1.380649 \times 10^{-23} \text{ (J/K)}$$

Monatomic Gas Model: When the molecular count in a container with an ideal gas is known (N), the heat capacity of the ideal gas is calculated quite well for Noble Gases using the equation:

$$Q = (3/2) N k_B \Delta T = [3N] [\frac{1}{2} k_B \Delta T] \quad (\text{J})$$

The above equation states that to raise a container of (N) Noble Gas molecules with some temperature increase (ΔT) requires a specific amount of heat input (Q). This analysis is used with good success for certain real gases that approximate ideal gases. Then ($3N$) the modes of freedom in the ideal gas allows estimation of energy density in a medium. In a gas state, all Noble gases exist as singular atoms termed monatomic molecules. The below table shows the success in applying the Equipartition Theorem to Noble gases (ref [65]), counting modes & assuming an equal distribution of energy among the monatomic molecules. Molar specific heat (c_p) is applied in the following empirical equation:

$$Q = c_p N \Delta T \quad (\text{J})$$

(N) is the molar number of monatomic molecules of a specific Noble gas, (ΔT) is the measured temperature change in the gas & (Q) is the total heat energy supplied to the Noble gas to achieve the temperature change. When specific heat (c_p) is expressed per collective weight of a Noble gas, specific heats vary. When the molar mass of the atom is considered, each monatomic molecule truly behaves as a “ping pong ball” readily

translating external energy into its own kinetic energy. Average speeds in different Noble gas elements vary to compensate for different elemental atomic weights (m_g).

Equipartition Theorem Validity for Noble Gases							
atomic element name	atomic sym	atomic number	vdW radius (pm)	specific heat (J/kg/K)	relative atomic mass (g/mol)	molar specific heat (J/mol/K)	error from average (%)
monatomic Noble gases							
Helium	He	2	140	5193	4.0	20.786	0.000%
Neon	Ne	10	154	1030	20.2	20.785	0.004%
Argon	Ar	18	188	520	39.9	20.786	0.001%
Krypton	Kr	36	202	248	83.8	20.786	0.001%
Xenon	Xe	54	216	158	131.3	20.786	0.002%
diatomic gases							
Hydrogen	H ₂	1	120	14300	1.0	14.414	5.5%
Nitrogen	N ₂	7	155	1040	14.0	14.567	4.5%
Oxygen	O ₂	8	152	919	16.0	14.703	3.6%
Flourine	F ₂	9	147	824	19.0	15.655	2.6%
Chlorine	Cl ₂	17	175	478	35.5	16.952	11.1%

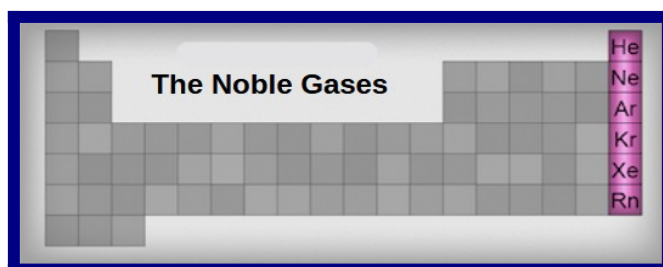
The above table lists van der Waals (vdW) radii of the listed elements in picometers (1 pm = 10⁻¹² meters). These radii indicate when molecule-specific QM 1st order corrections are valid for the “point particle” based ideal gas law. Then, “attractive and repulsive forces are considered” from charge distributions within each atom (ref [5]).

The Avogadro Constant is a count of molecules ([si4x6.pdf](#)) set @:

$$N_A = 6.02214076 \times 10^{23} \text{ (1/mol)}$$

This atomic count helps define the relative atomic mass of an element. The Avogadro Number is tied to the element carbon such that the Avogadro count of Carbon-12 atoms has about 12 grams mass. Other atomic weights @ the same atomic count (N_A) reflect a specific element’s stable isotope compositions with its relative atomic masses (m_g).

Diatomic “Ping Pong Balls”: The Noble gases occupy the last column on the Periodic Table (ref [66]). That last column indicates that all electron shells are completely filled & those e-shell configurations are very stable. Other elemental gases to the left of the last column have less stable incomplete electron shells; some form non-symmetric paired or diatomic molecules. Characteristics of these molecular gases can not be readily approximated by the Equipartition Theorem. They have other degrees of freedom (rotations, vibrations) that need to be considered.



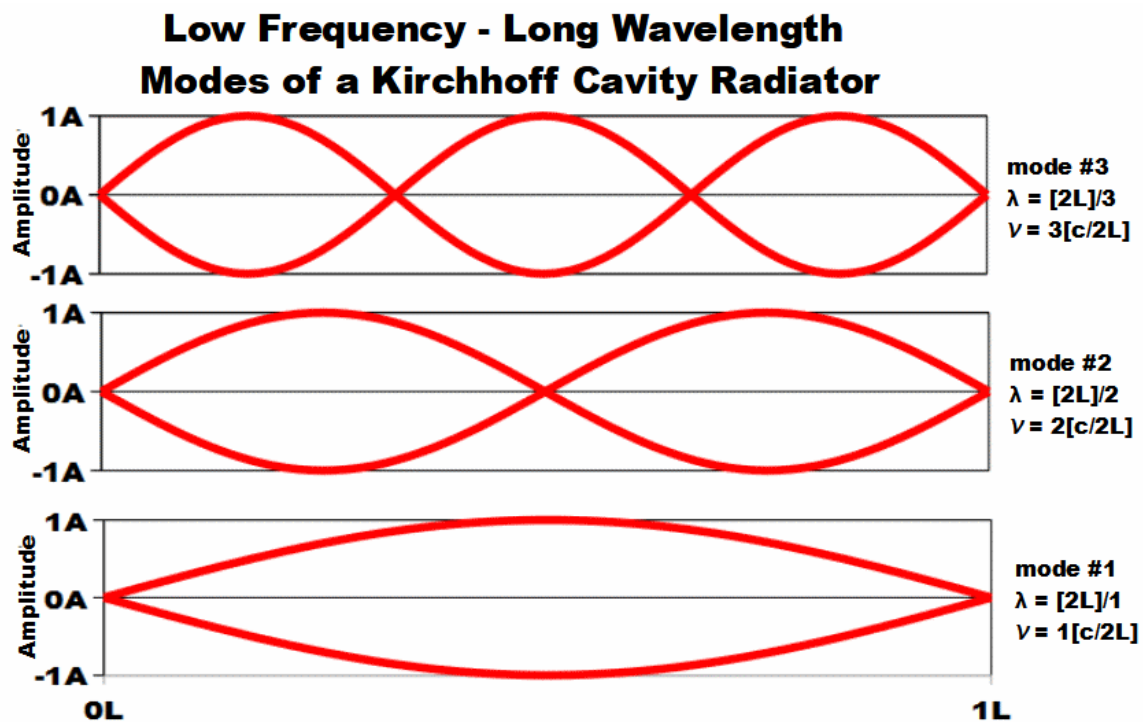
Monatomic “Ping Pong Ball” Collisions: The collisions of the Noble gas molecules with their glass container walls is completely elastic (like a “ping pong ball”), without

kinetic energy loss & generating no heat. In all atomic elements, when a molecule has a full complement of electrons-proton pairs, the molecule carries a net neutral charge. The key is the negative electron charge region is distributed around a much smaller nucleus of extremely dense positive protons.

$$F_{Coulomb}(\Delta r) = [1/4\pi\epsilon_0][q_1q_2]/(\Delta r \cdot \Delta r) \text{ (N)} \quad \text{Coulomb's Law (ref [67])}$$

As a neutral molecule approaches a neutral amorphous glass wall, the local negatively charge distribution of the gas molecule interfaces with that of the glass silicate below its van der Waals atomic radius. Quantum restrictions apply, but Coulomb magnitude forces between the two like charge distributions repel each other in an equal but opposite inverse distance-squared ($1/\Delta r^2$) manner. The component of molecular velocity normal to the glass wall reverses when the repelling forces are great enough. However, the Noble gas molecule *never* contacts the glass wall.

Planck wanted to use the same analysis for photon oscillators replacing translational molecules in his Kirchhoff radiation cavity. As we shall see, the heat capacity of photon oscillators in a Kirchhoff cavity radiator appears infinite unless QM assumptions are imposed.



Oscillator Modes: Planck used the Equipartition Theorem from Statistical Mechanics to develop his spectral energy density within the Kirchhoff cavity radiator. The radiator is assumed to be in a steady-state thermodynamic equilibrium & its temperature is stabilized over a lengthy duration. The photons bounce back & forth between the cavity walls & eventually settle into discrete oscillator modes determined by a characteristic length (L_{cavity}) for low frequencies & large wavelengths. In counting 1D oscillator modes

of a cavity with characteristic dimension (L_{cavity}), the modes increase as incremental frequencies ($m\nu$, $m = 1, 2, 3, \dots$) when steady-state reflection from cavity walls is accounted for.

To describe waves in a 3D space of mode (m) within the cavity, the spatial vector (\mathbf{k}_m) & temporal frequency (ω_m) are introduced. For a peak-to-peak wave cycle, ($1 \text{ cycle} = 360^\circ = 2\pi \text{ radians}$), then:

$$A_m(\mathbf{x}, t) = A_m \sin(\mathbf{k}_m \cdot \mathbf{x} + \omega_m t) \quad \text{with} \quad \mathbf{k}_m \equiv (2\pi / \lambda_m) \hat{\mathbf{n}}_m \quad \& \quad \omega_m \equiv 2\pi \nu_m$$

Recall for the Kirchhoff radiator cavity,

$$c \equiv \lambda_m \nu_m \quad \Leftrightarrow \quad \lambda_m = 2L / m \quad \& \quad \nu_m = m (c / 2L) \quad \text{with} \quad m = 1, 2, 3, \dots$$

The spatial vector (\mathbf{k}_m) is a 3D vector, then:

$$\mathbf{k}_m = 2\pi (m_r / 2L) \hat{\mathbf{n}}_m = (\pi / L)(m_r \hat{\mathbf{n}}_r) = (\pi / L) \mathbf{m}_r$$

$$(k_m)^2 \equiv (\mathbf{k}_m \cdot \mathbf{k}_m) = (\pi / L)^2 (m_x^2 + m_y^2 + m_z^2) = (\pi / L)^2 m_r^2$$

Note that the components of 3D mode vector (m_x, m_y, m_z) are always positive. Then, the volume of interest is the positive quadrant in 3D mode space. To count the number of modes (M) in a volume of modes locations (\mathbf{m}_r), for a sphere of mode radius (m_r) in 3D mode location space, the count is confined to the positive quadrant or 1/8 the sphere. In addition, an oscillator has a two-dimensional orientation once its one-dimensional propagation direction is specified. Therefore, our total mode count equation is multiplied by a factor of 2,

$$dM = 2 \left[\frac{1}{8} \cdot (4\pi m_r^2 dm_r) \right] \quad \text{with} \quad \nu_r = (c/2L_{cavity}) m_r \quad \Leftrightarrow \quad m_r = (2L_{cavity}/c) \nu_r$$

$$dm_r = (2L_{cavity}/c) d\nu_r \quad \Leftrightarrow \quad dM = \pi \left[2L_{cavity} \nu_r / c \right]^2 \cdot \left[2L_{cavity} / c \right] d\nu_r$$

$$dM = 8\pi \left[L_{cavity} / c \right]^3 \nu_r^2 d\nu_r$$

The Kirchhoff cavity volume [$V = (L_{cavity})^3$] is defined by the characteristic length (L_{cavity}). Then mode density (u_m) per unit volume in mode space is:

$$dM = \left[8\pi V / c^3 \right] \nu_r^2 d\nu_r \quad \Leftrightarrow \quad du_m = dM / V = \left[8\pi / c^3 \right] \nu_r^2 d\nu_r$$

Average Energy per Mode: The Equipartition Theorem that is being applied here, assumes that thermodynamic equilibrium has been reached. This equilibrium occurs when a sufficient time in a stable environment passes whereby energy between modes is exchanged until all modes within an equilibrium system have the same energy ($\bar{E}_{m,\nu}$). We can estimate a density of modes (u_m) within Kirchhoff's radiation cavity, multiply this by average energy per mode ($\bar{E}_{m,\nu}$) & we will get energy density ($w_{e,\nu}$) within the hollow cavity radiator.

$$w_{e,\nu} = \bar{E}_{m,\nu} u_m \quad \Leftrightarrow \quad dw_{e,\nu} = \bar{E}_{m,\nu} du_m = \left[8\pi \bar{E}_{m,\nu} / c^3 \right] \nu_r^2 d\nu_r \quad (\text{J/m}^3)$$

For a harmonic oscillator, its averaged energy is given by a spring potential & momentum averages (ref [68]):

$$F_{m,spring} = -a |\Delta x| \quad \Leftrightarrow \quad \bar{E}_{m,spring} = \langle \frac{1}{2} a (\Delta x)^2 \rangle_{avg} = \frac{1}{2} k_B T \quad \& \quad \bar{E}_{m,KE} = \langle \frac{1}{2} m_g v^2 \rangle_{avg} = \frac{1}{2} k_B T$$

$$\bar{E}_{m,total} = \bar{E}_{m,spring} + \bar{E}_{m,KE} = \frac{1}{2} k_B T + \frac{1}{2} k_B T = k_B T \quad (\text{J})$$

Ultraviolet Catastrophe: Adding absolute Temperature (T) to the radiation density equation:

$$dw_{e,\nu} = [8 \pi k_B T / c^3] \nu_r^2 d\nu_r \quad \Leftrightarrow \quad w_e = \int dw_{e,\nu} = [8 \pi k_B T / c^3] \int \nu_r^2 d\nu_r \quad (\text{W/m}^3)$$

$$w_e \propto \int \nu_r^2 d\nu_r \quad \text{with} \quad [\nu: 0 \rightarrow \infty] \quad \Leftrightarrow \quad w_e \rightarrow \infty$$

The above integral shows the “Ultraviolet Catastrophe”. For the energy per unit volume ($w_{e,total}$) in the Kirchhoff radiator cavity, increasing frequencies with smaller wavelengths will continue to absorb equal amounts of energy, allowing the Kirchhoff cavity radiation density ($w_{e,total}$) an infinite energy storage.

During the turn of the last century, several distinguished Physicists tried to formulate a model deriving measurable spectral Kirchhoff cavity exitance ($M_{e,\nu}(T)$). In 1900, Lord Rayleigh (1842 – 1919) (ref [69]) & Sir James Jeans (1877 – 1946) (ref [70]) used the Equipartition Theorem to derive the above spectral energy density with an infinite evaluation. Rayleigh believed the Equipartition Theorem failed “to be valid for high-frequency vibrations.” Jeans argued that possibly a “luminous aether may not be in thermal equilibrium.” (ref [71])

The Boltzmann’s Distribution – A Quantum True Start

While learned minds of the day discussed the “Ultraviolet Catastrophe”, Max Planck forged ahead & developed his “quantum solution” to the “catastrophe” problem. Planck had several temperature dependent spectral data sets of Kirchhoff radiators in the visible range. His task was to develop assumptions that derived constants mathematically to fit the data. Planck kept the same modes (m) & frequencies (ν_m) as determined by the cavity’s characteristic length (L_{cavity}). The Equipartition Theorem assumed an average energy per mode ($k_B T$) & an equal probability of this energy per mode distributed equally across all oscillator modes.

Changing this requirement, Planck imposed a minimum energy of each oscillator proportional to each frequency ($E_1(\nu_m) \equiv h\nu_m$). The quantum constant (h) would be determined by experiment. Planck also imposed discrete (n) energy amplitudes ($E_{nm} \equiv nE_1(\nu_m)$) for each oscillator mode (m) & frequency (ν_m). Instead of equal likelihood across all oscillator frequency modes, the Boltzmann distribution gives a decreasing probability of energy states with increasing energy (E_{nm}) & frequency (ν_m) (ref [72]). This distribution weights probabilities toward lower energies & is derived to maximize a system’s entropy (see below).

$$E_{nm} = E_n(\nu_m) = nE_1(\nu_m) = nh\nu_m = n [m c h / 2 L_{cavity}] \quad (\text{J})$$

For lower probabilities, a fixed overall energy allocated into fewer high energy states implies more order & lower entropy. That same energy allocated into more low energy states implies more disorder & higher entropy; these states are given higher likelihoods.

Incremental Entropy & Energy: The Boltzmann probability distribution prescribes a

fixed total energy among Boltzmann's enumerated states of possible energy storage within a thermodynamic system in equilibrium. An overall thermodynamic system entropy (S) is related to number of distinct available microscopic energy states (W) by the equation (ref [73]):

$$S = k_B \ln(W) \quad (\text{J/K})$$

Entropy has the units of (J/K – Energy / Temperature). The units reflect the 2nd Law of Thermodynamics. Here, the 2nd Law states anytime heat flows within a closed system, the summation of heat flow divided by the absolute temperature of each heat exchange (Q_i / T_i) always increases the overall entropy (S).

$$\Delta S = \sum (Q_i / T_i) > 0 \quad (\text{J/K})$$

The (Boltzmann Constant) \times (Thermodynamic Temperature) product ($k_B T$) is a characteristic scale value for all discrete members of the statistical ensemble being evaluated. Recall for the previous monatomic gas example, the molecular average kinetic energy of every Noble gas atom was given by ($\frac{1}{2}k_B T = \langle \frac{1}{2}m_g v^2 \rangle_{avg}$). For the cavity radiator using the Boltzmann distribution, the factor ($\frac{1}{2}k_B T$) represents the averaged oscillator energy among the cavity oscillators (ref [74]).

Planck chose quantized amplitude increments for each oscillator frequency to have a finite number of microscopic energy states (W). Otherwise, the evaluated entropy (S) would be infinite (ref [73]). Absolute temperature (T) is the equilibrium temperature defined for the thermodynamic state of the cavity radiator.

Boltzmann Probability Characteristics: The Boltzmann's probability distribution allocates more energy to lower energy states (ref [72]). For ($p(E_1)$ & $p(E_2)$) with ($E_1 < E_2$), Boltzmann required that ($p(E_1) > p(E_2)$). Require:

$$p(E_2) / p(E_1) < 1 \quad \text{for} \quad E_1 < E_2$$

For the scaled energy exponential:

$$\lim_{E \rightarrow 0} e^{-E/k_B T} = 1 \quad \& \quad \lim_{E \rightarrow \infty} e^{-E/k_B T} = 0$$

$$\ln [p(E_2) / p(E_1)] < 0 \quad \Leftrightarrow \quad (E_1 - E_2) / k_B T < 0 \quad \Leftrightarrow \quad -(E_2 - E_1) / k_B T < 0$$

$$\ln [p(E_2) / p(E_1)] = -(E_2 - E_1) / k_B T \quad \Leftrightarrow \quad p(E_2) / p(E_1) = \exp [-(E_2 - E_1) / k_B T]$$

Boltzmann's probability that energy mode (i) will have energy state (E_i) is:

$$p(E_i) = \frac{\exp(-E_i/k_B T)}{\sum_{j=0}^n \exp(-E_j/k_B T)}$$

The presentation given here shows that mode (i) with ($p(E_i)$) will have greater probability with lower energy (E_i). Lagrangian Multipliers can be used for a more formal proof (ref [74]) to show the Boltzmann distribution gives a stable, average energy distribution for a thermodynamic system.

Ultraviolet Catastrophe Averted: Recall, Planck imposed a “quantized” energy for the oscillators in a Kirchhoff cavity radiator essentially allocating discrete energy amplitudes for the oscillators. High frequency oscillators in the ultraviolet range with wavelengths approaching atomic dimensions would have higher energies. Per the Boltzmann distribution, these high energy states would have lower probabilities.

Oscillator Energy Probabilities: The erroneous Equipartition Theorem assigned equal energies to all frequency modes with continuous oscillator amplitudes. The Boltzmann distribution assigned higher probability to lower discrete (m) oscillator energies & the energy amplitudes had discrete incremental (n) values.

$$E_n(\nu_m) = nh\nu_m \quad \text{with} \quad \nu_m = m(c / 2 L_{\text{cavity}}) \quad \& \quad \lambda_m = (2 L_{\text{cavity}}) / m \quad \text{with} \quad m, n = 1, 2, 3 \dots$$

$$\bar{E}_m(\nu_m) \equiv \sum_{n=0}^{\infty} p(n) E_n(\nu_m) = \frac{\sum_{n=0}^{\infty} E_n(\nu_m) \exp(-E_n(\nu_m)/k_B T)}{\sum_{n=0}^{\infty} \exp(-E_n(\nu_m)/k_B T)} = \frac{\sum_{n=0}^{\infty} nh\nu_m \exp(-nh\nu_m/k_B T)}{\sum_{n=0}^{\infty} \exp(-nh\nu_m/k_B T)}$$

Let,

$$x \equiv \exp(-h\nu_m/k_B T) \quad \Leftrightarrow \quad \bar{E}_m(\nu_m) = \frac{\sum_{n=0}^{\infty} nh\nu_m x^n}{\sum_{n=0}^{\infty} x^n} = h\nu_m \left(\frac{\sum_{n=0}^{\infty} n x^n}{\sum_{n=0}^{\infty} x^n} \right)$$

$$\bar{E}_m(\nu_m) = h\nu_m \frac{(x + 2x^2 + 3x^3 + \dots)}{(1 + x + x^2 + x^3 + \dots)} = h\nu_m \frac{x(1 + 2x + 3x^2 + 4x^3 + \dots)}{(1 + x + x^2 + x^3 + \dots)}$$

These infinite sums can be simplified through geometric series identities (ref [75]).

$$\frac{1}{(1-x)} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n = 1 + 2x + 3x^2 + 4x^3 + \dots$$

Substituting into the average energy of each frequency mode (m) is

$$\bar{E}_m(\nu_m) = (h\nu_m x) [(1-x)/(1-x)^2] \quad \text{with} \quad \nu_m = m(c / 2 L_{\text{cavity}})$$

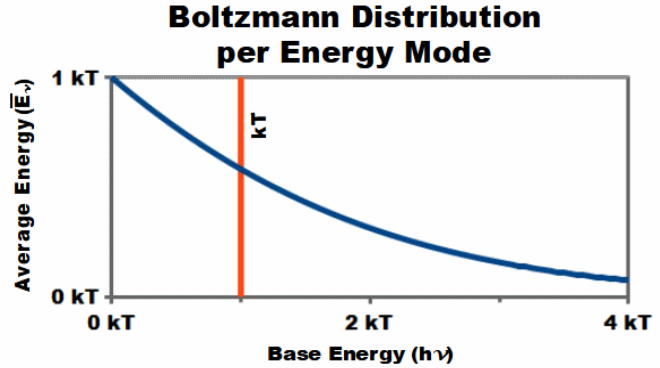
$$\bar{E}_m(\nu_m) = (h\nu_m x) / (1-x)$$

$$\bar{E}_m(\nu_m) = (h\nu_m) / (1/x - 1)$$

$$\bar{E}_m(\nu_m) = (h\nu_m) / [\exp(h\nu_m/k_B T) - 1]$$

The graph below plots the average energy per frequency mode (\bar{E}_m) versus the base frequency mode energy ($h\nu_1$). Both the average energy (\bar{E}_m) & frequency mode energy ($h\nu_m$) are scaled through the incremental Boltzmann Energy ($k_B T$).

The graph shows as the base energy ($h\nu_1$) falls below the Boltzmann Energy ($k_B T$), the average Energy (\bar{E}_m) approaches ($k_B T$) the Equipartition Theorem Energy. A Kirchhoff cavity radiator with characteristic dimension (L) has a lowest cutoff frequency ($\nu_{low} = c / 2L_{cavity}$). Then, as the Kirchhoff cavity volume increases, the lowest average Energy ($\bar{E}_{m,\nu}$) approaches the “classical limit” Equipartition Theorem Energy ($k_B T$). As required, the Boltzmann Distribution allocates high probabilities to low base energy levels.



Recall the spectral radiation density inside the Kirchhoff cavity.

$$dw_{e,\nu} = [8\pi \bar{E}_m / c^3] (\nu_m)^2 d\nu_m = [8\pi h\nu_m / c^3] / [\exp(h\nu_m / k_B T) - 1] (\nu_m)^2 d\nu_m$$

Derived Kirchhoff Spectral Coefficients: Substituting in the derived constants, with ($m = 1, 2, 3, \dots$), spectral energy density is:

$$w_{e,\nu}(T) d\nu_m = [8\pi h (\nu_m / c)^3] / [\exp(h\nu_m / k_B T) - 1] (\nu_m)^2 d\nu_m \quad \text{with } \nu_m = m \cdot [c / 2 L_{cavity}]$$

$$w_{e,\lambda}(T) d\lambda_m = [8\pi hc / (\lambda_m)^5] / [\exp(hc / \lambda_m k_B T) - 1] d\lambda_m \quad \text{with } \lambda_m = [2 L_{cavity}] / m$$

For a sufficiently large (L_{cavity}), integration replaces summation & the spectral exitance ($M_{e,\nu}(T)$) from the Kirchhoff cavity radiator is:

$$M_{e,\nu}(T) d\nu = (c/4) \cdot w_{e,\nu}(T) d\nu = [2\pi h / c^2] / [\exp(h\nu / k_B T) - 1] \nu^3 d\nu \quad (\text{W/m}^2)$$

$$M_{e,\lambda}(T) d\lambda = (c/4) \cdot w_{e,\lambda}(T) d\lambda = [2\pi hc^2] / [\exp(hc / \lambda k_B T) - 1] \lambda^{-5} d\lambda \quad (\text{W/m}^2)$$

The derived coefficients of the above equations agree well with experimental spectral data for an effective Solar Kirchhoff cavity exitance temperature 5772 K.

Stefan–Boltzmann Law Derived: The law states total Kirchhoff cavity exitance ($M_e(T)$) varies as (T^4) (ref [61]). Then, the Kirchhoff cavity spectral exitance ($M_{e,\nu}(T)$) should be integrated over all frequencies to match this empirical results.

$$M_e(T) = \int M_{e,\nu}(T) d\nu = [2\pi h / c^2] \int \nu^3 / [\exp(h\nu / k_B T) - 1] d\nu \quad (\text{W/m}^2)$$

with $[\nu: 0 \rightarrow \infty]$

From integral Calculus, let:

$$x \equiv h\nu / k_B T \quad \Leftrightarrow \quad \nu = (k_B T / h) x \quad \Leftrightarrow \quad d\nu = (k_B T / h) dx$$

$$M_e(T) = [2\pi h / c^2] [k_B T / h]^4 \int [x^3 / (\exp(x) - 1)] dx = [2\pi k_B^4 / c^2 h^3] [T^4] \int [x^3 / (\exp(x) - 1)] dx$$

with $[x: 0 \rightarrow \infty]$

When both lower & upper limits of an integral are specified, the integral is termed a “definite integral”. BTW, if the limits are **not** specified, the nomenclature in consulting

integral tables is an “indefinite integral”. From integral tables (ref [62] & ref [75]),

$$\int [x^3 / (\exp(x) - 1)] dx = \pi^4/15 \quad \text{with} \quad [x: 0 \rightarrow \infty]$$

Substituting, the radiation power per square meter @ ideal 100% emissivity (ϵ) is:

$$M_e(T) = [2\pi k_B^4 / c^2 h^3] [\pi^4 / 15] [T^4] = [2\pi^5 k_B^4 / 15 c^2 h^3] [T^4] \quad (\text{W/m}^2)$$

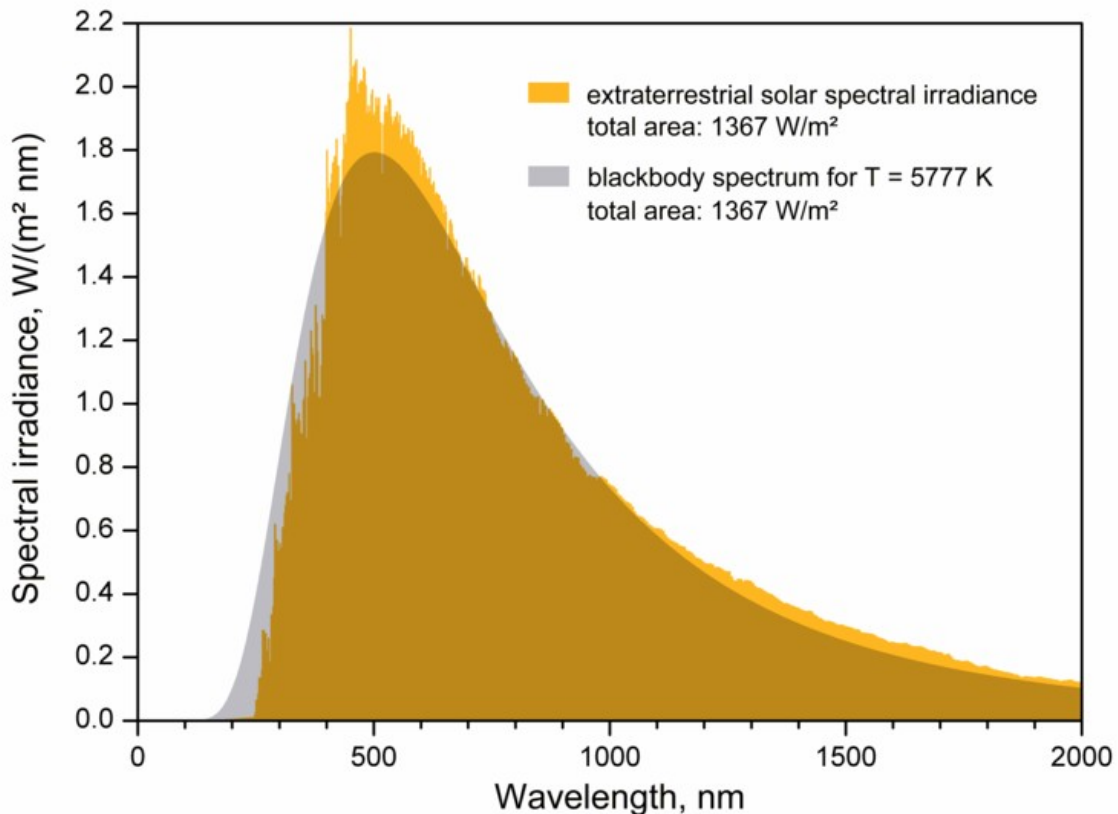
In the ideal condition, all incident radiation is absorbed ($\alpha = 100\%$) & re-radiated ($\epsilon = 100\%$) @ spectral Kirchhoff cavity exitance ($M_{e,\nu}(T)$) values. For realistic absorption & emissivity conditions ($\alpha < 100\%$, $\epsilon < 100\%$):

$$M_e(T) = \epsilon \sigma T^4$$

$$\sigma \equiv [2\pi^5 k_B^4 / 15 c^2 h^3] = 5.670374419 \times 10^{-8} \quad (\text{W/m}^2/\text{K}^4)$$

The Stefan-Boltzmann Constant (σ) is readily confirmed experimentally ([si4x6.pdf](#)) & verifies the quantum assumptions of Max Planck. For total Solar irradiance received @ the Earth, the same Sun’s radius / Earth’s mean orbit ratio squared is applicable. When total energy density ($w_e(T)$) within the Kirchhoff cavity radiator are calculated based on exitance ($M_e(T)$) from absolute Temperature (T):

$$w_e(T) = [4\sigma/c] T^4 = [4/\epsilon c] \cdot M_e(T) \quad (\text{J/m}^3)$$



The greater spectrum of the Sun as measured above Earth’s atmosphere shows the

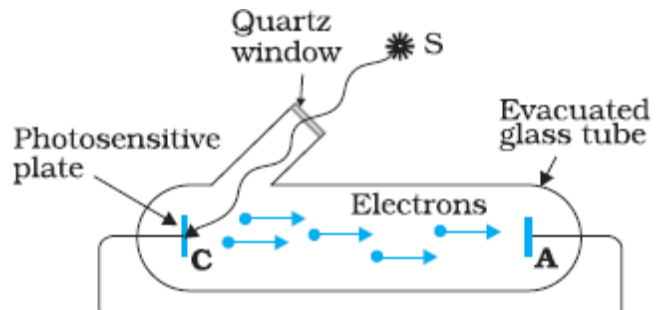
good approximation of the Solar surface radiance as a spectral Kirchhoff cavity radiator (ref [76]). When the coefficients of the theoretical spectral Kirchhoff radiance are fully known, a Solar effective temperature can be calculated that best matches the Sun's measured spectral irradiance data @ the Earth.

The “Soon to be Great” Max Planck published his quantum hypothesis in 1900, but it was considered a “fringe” theory, until another “Soon to be Great” Physicist used the same quantized photon idea to explain another experiment, 5 years later.

Validation – Photoelectric Effect: Around the turn of the century, Physicists performed an experiment directing a monochrome light beam @ a metal plate in a vacuum. They then measured the kinetic energy of the free electrons ejected from the metal plate (refs [77] & [78]). Three parameters were adjusted in the experiment:

- wavelength of the incident light.
- intensity of incident light.
- Voltage between the plate cathode & a detector anode.

The image to the right shows the experimental setup (ref [79]). The light source (**S**) directs monochromatic light @ a specific frequency toward a metal plate cathode (**C**).



Filtered ultraviolet light was typically used on a metal plate absent of surface oxides or impurities. Dislodged electrons make their way to the anode (**A**) within a partial vacuum. Voltage can be adjusted between (**C**) & (**A**) to record “stopping voltage” that measures kinetic energy of the dislodged electrons. The below data graphs assume constant incident light intensity. From the experimental setup, a few observations were made:

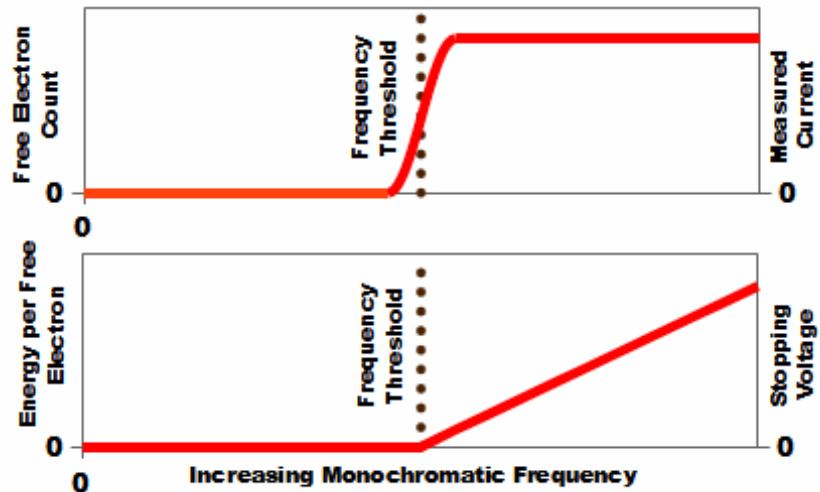
- #1 Below a certain monochromatic frequency, no light dislodges electrons regardless of intensity of the incident beam.
- #2 Once a frequency threshold is reached, increased light frequency does not increase the number of dislodged electrons.
- #3 Once a frequency threshold is reached, the “stopping voltage” of dislodged electrons increases with increased incident light frequency.

In 1905, along with his paper on Special Relativity (SR), [Albert Einstein](#) published a paper explaining the unusual electron behavior in the photoelectric effect (ref [80]). Einstein assumed a “quantized” energy of photons just as Max Planck had done for Kirchhoff Radiance, 5 years earlier. Einstein explained the discrete energy states of the Kirchhoff cavity radiator by identifying the wall atoms. These walls are limited to emit the reflected radiation in a quantized manner (ref [81]).

$$E_n(\nu) = nh\nu \quad \text{with } n = 1, 2, 3, \dots$$

The noted frequency (ν) threshold (#1, #2, #3) reflects the base energy level ($E_f(\nu)$) of the photon from the monochromatic light, in part (as seen in both data plots). As the mono-chromatic light frequency increases, dislodged electrons (electric current) remain constant (#2) (as seen in the “current” plot above), but individual electrons absorb a greater photon energy (#3) contained within the incident beam (as seen in the “voltage” plot).

Photoelectric Effect Experimental Results



The photoelectric effect & Einstein's verification of quantum photons helped elevate the Quantum World as a possible solution in Modern Physics. In turn, Planck promoted Einstein's Theory of SR. Along with General Relativity (GR), Einstein was propelled to celebrity status when GR was “proved” in a 1919 solar eclipse (ref [82]). Yet Quantum Theory languished. In 1950, a jaded Dr. Planck observed, “A new scientific truth does not triumph by convincing its opponents ... but rather because its opponents eventually die...” (ref [83]).

When a QM Oracle says it's True: Almost my entire Mechanical Engineering (ME) undergraduate degree studied a Classical “Ping Pong Ball” World. Obtaining a graduate degree in Physics, I learned QM with my imposed caveat, one day “hidden variables” may help clear up its more incredible aspects. Then, I learned about Bell's Inequality, studied it & accepted it. I don't understand a lot of QM, but now I have completely consumed my mug of QM Kool-Aid along with a “dizzy” hangover Niels Bohr may have alluded to!

With my drink of Kool-Aid, I relinquished my quest to know a reality to the detail I sought. At Princeton walking with his colleague Abraham Pais when the Moon was out, Einstein asked Pais, “Do you really believe the Moon is not there when you are not looking at it?” (ref [84]) My common sense “Ping Pong Ball” World is built on a QM World whose behavior is mostly foreign to my experience. I am left with the data points that imply two disconnected models, the Classical & QM models, which are in the pantheon of valid & successful models of Physics. Considering QM applications created the computer electronics that allow me to compose this document, QM of modern Physics is quite adequate!

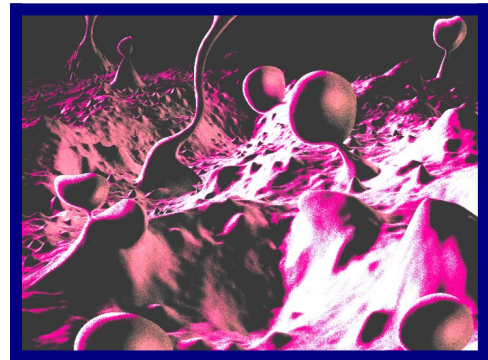
Conclusion

A Reason for a Planck Constant: The Equipartition Theorem was applied unsuccessfully to spectral Kirchhoff cavity radiation. That method had been used accurately in other areas of Statistical Mechanics. As a result, Physicists of the day cogitated on that conundrum for so long that Paul Ehrenfest labeled it “the Ultraviolet Catastrophe” in 1911 (ref [85]). Planck’s Constant has appeared in many other equations in the Quantum World since 1900. However, Max Planck was the 1st to find a need for it.

The Wrong Probability Distribution: When encountering the incorrect infinite energy storage of a Kirchhoff cavity radiator, the wrong probability model was being used, i.e., the Equipartition Theorem. Planck developed an equation form first, then adopted Boltzmann’s probability model with an added quantum assumption to derive his curve-fit form. The Boltzmann’s distribution assigned near zero probabilities @ high frequencies in conjunction with relatively large quantum energies. With these changes, Planck derived Kirchhoff spectral radiance accurately.

Quantum Foam: Both of Planck’s models subdivide the cavity oscillator wavelengths into infinitesimal measures & large, energetic amplitudes in the frequency limit approaching infinity. Not conjectured by 1900 Physicists, recent quantum theorists assert an orderly smooth space-time should exist within a thousandths of the diameter of a proton (refs [86] & [87]). A proton radius is about 0.8 femtometer ($1 \text{ fm} = 10^{-15}$ meters) or nicknamed a “Fermi”, after Enrico Fermi (1901 – 1954) (ref [88]). Divide a Fermi by a thousand & the unit of measure is an attometer ($1 \text{ am} = 10^{-18}$ meters).

Smaller than an attometer, down through the Planck length of 10^{-35} meters & smaller, a disruptive space-time is hypothesized to be anything but empty space. Heisenberg’s Uncertainty Principle may be in full effect. A zoo of subatomic particle & their anti-matter particles (pions, kaons, neutrinos, electrons, etc.) are theorized to pop into existence & quickly vanish. These diametric subatomic pairs borrow their energies from the future, then collapse again into nothing returning their energies to the past as if nothing happened. Even extra dimensions may appear, then disappear. An artist captured the idea of Quantum Foam in the illustration above(ref [89]). Note, our visible light wavelengths in nanometers are billions times longer than the quantum foam scale & can not possibly discern any detail. An illumination @ foam dimensions, photons take on energies a billion times greater & may possibly obliterate what one is trying to observe.



Extrapolation into the Unknown: Both the Equipartition Theorem & Boltzmann Distribution try to assess meaningful photon wavelengths & frequencies overlaid on this hypothesized foam. Quantum Foam may exist on scales of Planck length @ 10^{-35} meters, across Planck time @ 10^{-44} seconds & in energies @ 10^{+9} Joules. “These scales are characteristic of the initial moments of the big bang.” (ref [89]) The

Boltzmann distribution works in this application because it assigns oscillator energies with underflow probabilities contributing a near zero integral across the foam. On the contrary, the Equipartition Theorem of the “Ultraviolet Catastrophe” assigns an equal probability of a ($k_B T$) energy to **all** oscillators on this quantum foam which is an extrapolation into the ultimately incorrect!

Far From Obsolete: The discrete assumptions of QM were not required in the Classical “Ping Pong Ball” World for 200 years after Isaac Newton 1st described it. Indeed, while @ University of Munich in 1878, a professor advised the young Max Planck not to go into Physics (ref [90]). The professor remarked, “Almost everything is already discovered, and all that remains is to fill a few unimportant holes.” Obviously, Max Planck did not follow his teacher’s advice. Today, legions of Scientists & Engineers dedicate their entire careers to filling in the canyon from the earthquake Max Planck caused in 1900.

References

- [1] Khan Academy, [Dalton's Atomic Theory](#), 2023.
- [2] LibreText.org, Chemistry, [Indivisible – The Atomic Theory](#), 2022.
- [3] Wikipedia.org, [Isaac Newton](#), 2023.
- [4] Bachter, Nikolai, Thinking about Thinking, [Newton's Classical Mechanics](#), 2017.
- [5] Wikipedia.org, [Van der Waals Equation](#), 2023.
- [6] Applied Mathematics, U of Waterloo, [Continuum & Fluid Mechanics](#), 2023.
- [7] Wikipedia.org, [Albert Einstein](#), 2023.
- [8] Nemo, John, PhysicsForums.com, [The Fallacy: Classical Physics is Wrong!](#), 2018.
- [9] Wikipedia.org, [Modern Physics](#), 2024.
- [10] Sutter, Paul, PhD, Space.com, [Do We Live in a Quantum World?](#), 2019.
- [11] Löffler, Max, ScienceNews.org, [Quantum Revolution – Nature of Reality](#), 2023.
- [12] Ornes, Stephen, SNExplores.org, [Quantum World – Mind-Bogglingly Weird!](#), 2017.
- [13] Wikipedia.org, [Niels Bohr](#), 2023.
- [14] Wikipedia.org, [Richard Feynman](#), 2023.
- [15] George, Everett, *Idea Contributions*, [Computer Chips Demystified](#), 2023.
- [16] Siegel, Ethan, BigThink.com, [Can Hidden Variables Save QM?](#), 2022.
- [17] Stadermann, Kirsten, et al, *IOP Science*, [Teaching Materials links QM](#), 2021.
- [18] Wikipedia.org, [Max Planck](#), 2023.
- [19] Alamy.com, Portrait – Max Plank, [Image ID: ERHFER](#), 2023.
- [20] Joseph, Kathy, KathyLovesPhysics.com, [Max Planck's Quantum Theory](#), 2019.
- [21] SpeechModification.com, [Sounds of the Letter 'A' in English](#), 2022.
- [22] Pronunciation Guide, YouTube.com, [Pronounce “Max Planck”](#), 2023.
- [23] Hewitt, Paul G, *Conceptual Physics*, 12th Ed, Pearson Press, 2015.
- [24] Ball, Phillip, Mètode Science Studies, [Einstein & Nazi Physics](#), 2020.
- [25] Wikipedia.org, [Erwin Planck](#), 2023.
- [26] French, AP, PhD, *Special Relativity*, 1st Ed, MIT Physics, WW Norton & Co, 1968.
- [27] Wikipedia.org, [Werner Heisenberg](#), 2023.
- [28] Joseph, Kathy, KathyLovesPhysics.com, [Werner Heisenberg Biography](#), 2020.
- [29] LibreText.org, Physical Chemistry, [Heisenberg's Uncertainty Principle](#), 2023.

- [30] Plato Philosophy, Stanford U, [Quantum Theory Uncertainty](#), 2016.
- [31] Harrison, David, U of Toronto, [Complementary, Copenhagen Interpretation](#), 2006.
- [32] Wikipedia.org, [Uncertainty Principle](#), 2023.
- [33] Buttar, Simran, Secrets of the Universe, [Dirac Equation & Anti-Matter](#), 2022.
- [34] Khan Academy, [Understanding the Wave – Particle Duality](#), 2023.
- [35] Feynman, RP, et al, Feynman Lectures on Physics, Caltech, [Volume III](#), 2013.
- [36] Wikipedia.org, [Double-Slit Experiment](#), 2023.
- [37] Wikipedia.org, [Quantum Decoherence](#), 2023.
- [38] Wikipedia.org, [Snell's Law](#), 2023.
- [39] Sutter, Paul, PhD, Space.com, [Speed of Light Properties Explained](#), 2020.
- [40] Semiconductors, LibreText.org, [Absorption of Light & Generation](#), 2021.
- [41] Paulson, Steve, U of Wisconsin, WPR.org, [Isaac Newton's Secret Alchemy](#), 2020.
- [42] Wikipedia.org, [Joseph von Fraunhofer](#), 2023.
- [43] Joseph, Kathy, KathyLovesPhysics.com, [Spectrosopy Changed the World](#), 2019.
- [44] Wikipedia.org, [Thomas Young](#), 2023.
- [45] Nave, CR, PhD, GA State, [HyperPhysics: Diffraction Gratings](#), 2017.
- [46] Khan Academy, [Double-Slit Pattern Explanation](#), 2023.
- [47] Wikipedia.org, [Gustav Kirchhoff](#), 2023.
- [48] Klus, Helen, PhD, the Star Garden, [Spectral Lines](#), 2017.
- [49] Wikipedia.org, [Radiant Exitance](#), 2023.
- [50] George, Everett, *Idea Contributions*, [Thermodynamics Demystified](#), 2024.
- [51] Helmenstine, AM, PhD, ThoughtCo, [Kelvin Temperature Scale Definition](#), 2020.
- [52] Wikipedia.org, [Emissivity](#), 2023.
- [53] Spakovszky, ZS, et al, MIT.edu, [Thermodynamics – Ideal Radiators](#), 2007.
- [54] Pharr, Matt, et al, *Physically Based Rendering*, [Radiometric Integrals](#), 2021.
- [55] Wikipedia.org, [Inverse Square Law](#), 2024.
- [56] Helmenstein, AM, PhD, ThoughtCo, [Understand the Visible Spectrum](#), 2020.
- [57] Wikipedia.org, [Effective Temperature](#), 2023.
- [58] Wikipedia.org, [Wien's Displacement Law](#), 2023.
- [59] OmniCalculator.com, [Wein's Law Calculator](#), 2022.
- [60] Rapid Tables, [Online Comprehensive Units Conversion Calculator](#)
- [61] Wikipedia.org, [Stefan–Boltzmann Law](#), 2023.
- [62] eDisciplinas.usp.br, U of São Paulo, [The Derivation of the Planck Formula](#), 2022.
- [63] LibreText.org, Physical Chemistry, [Equipartition Theorem](#), 2023.
- [64] U of Oxford, Chemistry, [Equipartition Theorem](#).
- [65] Gray, Theodore, et al, Wolfram Research, [The Periodic Table](#), 2017.
- [66] Shmoop U, the Periodic Table, [the Noble Gases](#), 2023.
- [67] Electrical 4 U, [Coulomb's Force Law](#), 2020.
- [68] Wikipedia.org, [Equipartition Theorem](#), 2023.
- [69] Wikipedia.org, [John William Strutt, Lord Rayleigh](#), 2023.
- [70] Wikipedia.org, [James Jeans](#), 2023.
- [71] Wikipedia.org, [Rayleigh – Jeans Law](#), 2023.
- [72] Wikipedia.org, [Boltzmann Distribution](#), 2023.
- [73] Joseph, Kathy, KathyLovesPhysics.com, [Boltzmann Entropy Equation](#), 2020.

- [74] Benjaminson, E, Carnegie Mellon U, [Proof of the Boltzmann Distribution](#), 2022.
- [75] Spiegel, MR, PhD, et al, [Schaum's Math Handbook](#), 5th Ed, McGraw-Hill, 2017.
- [76] Iqbal, Mohammad, [An Introduction to Solar Radiation](#), Academic Press, 1983.
- [77] Wikipedia.org, [Photoelectric Effect](#), 2023.
- [78] Khan Academy, [Photoelectric Effect](#), 2023.
- [79] Chourasiya, Samar, MyPhysicsTurtorial, [Dual Nature of Photons & Matter](#), 2023.
- [80] Joseph, Kathy, KathyLovesPhysics.com, [The Photoelectric Effect & Einstein](#), 2019.
- [81] Halliday, David, et al, [Principles of Physics](#), 11th Ed, Wiley, 2020.
- [82] Powell, Devin, DiscoverMagazine, [The Eclipse That Made Einstein Famous](#), 2019.
- [83] Chawla, DS, ChemistryWorld, [Science Advances One Funeral @ a Time](#), 2019.
- [84] Quotes, Einstein, GoodReads.com, [Einstein's Moon Quote](#), 2023.
- [85] Wikipedia.org, [the Ultraviolet Catastrophe](#), 2023.
- [86] Bolles, Dana, NASA.gov, [Quantum Foam](#), 2015.
- [87] Wikipedia.org, [Quantum Foam](#), 2023.
- [88] Wikipedia.org, [Enrico Fermi](#), 2023.
- [89] Lindley, David, Physics, APS.org, ["Quantum Foam" Scrubs Away Energy](#), 2019.
- [90] FamousScientsts.org, Physicists, [Max Planck](#), 2023.

Note: Please donate \$50/year if you find [Wikipedia](#) useful!!!
Please donate \$50/year if you find [Khan Academy](#) useful!!!

Everett George 16 August 2024
Copyright © 2024 [jefgeorge.com](#)

Appendix: Spectral Kirchhoff Radiation

This article replaces nomenclature proposed by Gustav Kirchhoff that he certainly did not mean to be hostile or offensive to **any** ethnic group. However, Kirchhoff proposed this nomenclature in 1862 when, half a world away, a great conflict occurred over the civil rights of the very ethnic group who may find the term offensive. That's just one more reason to replace the terminology!

Offensive nomenclature that is unclear to the public can be construed to mean something entirely different as conveyed by an unscrupulousness press (ref [1]). Then, the message, academia is trying to portray to the public, is deflected through the very tools it uses to describe the message. I use "Kirchhoff radiator" with no loss in meaning or understanding in this article. Physics should replace any potentially offensive nomenclature **pre-emptively** now & in the future.

References

[1] George, Everett, *Idea Contributions*, [Black / Gray / White Body Radiator](#), 2023.