Preliminary

Dahlgren Vector Equations of Special Relativity

A. Introduction

The equations in this document modify Newtonian physics for the primary postulates (ref [7]) of Special Relativity (SR):

1) The speed of light in a vacuum c is measured the same regardless of its source.
2) All laws of nature are independent of a reference frame's constant velocity $\boldsymbol{v}$.

When a quantity is transformed to a reference frame with a velocity $\boldsymbol{v}$ at or less than c:

a) all time t derivatives of the equations hold the transformation velocity $\boldsymbol{v}$ constant.
b) all transformed quantities will reflect traveling at or less than c.
c) all information will travel at or less than c.
d) rest mass m of a particle is measured at rest time τ with particle velocity $\boldsymbol{u}$=0.
e) distance difference ($\boldsymbol{\Delta r}$) transformations assume constant velocity ($\boldsymbol{u}_t$).

B. MKSA Units

Meter-Kilogram-Second-Ampere (MKSA) units are used in this document, exclusively. The light speed in a vacuum (c), a fundamental constant, is defined an exact value (ref [9]). To measure Electric ($\boldsymbol{E}$) fields with Farad's (F) & Magnetic ($\boldsymbol{B}$) fields with Henry's (H), the following permittivity constant & permeability constant are defined in MKSA units (ref [12]):

$c \equiv 299\,792\,458$ m/s – speed of light (m/s)

$\mu_0 \equiv 4\pi \times 10^{-7}$ (H/m) – permeability constant (N/A^2)

$\epsilon_0 = 8.854\,187\,8128 \times 10^{-12}$ (F/m) – permittivity constant (s A)^2/(N m^2)

$\epsilon_0 \mu_0 = 1/c^2$

C. The Speed of Light – *c*

The speed of light (c) serves as four distinct functions in these equations:

a) a conversion factor between time & space in 4D space-time, ($\boldsymbol{r}^2 - c^2 t^2$ = Constant)
b) a conversion factor between mass & energy, ($E=mc^2$).
c) a conversion factor between electromagnetic (EM) values, ($\boldsymbol{B}=(\boldsymbol{v}\times\boldsymbol{E})/c^2$).
d) a unitless ratio limit for all speeds & velocities, ($\gamma^2=1/(1-\boldsymbol{v}^2/c^2)$).

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The metric system should have abolished different units of measure. Unfortunately, other "metric" units for smaller scales exists, e.g., cgs units (ref [3]). In addition, higher forms of elegance in expressing SR equations have appeared. For example, the *c's* (speed of light) can be eliminated from the equations contained herein completely.

D. The Equations

§1 Miscellaneous Equations

§1.1 Auxiliary Variables (ref [4])

$$\gamma_v \equiv \frac{1}{\sqrt{1-\mathbf{v}^2/c^2}} \qquad \delta_{uv} \equiv \frac{1-(\mathbf{u}\cdot\mathbf{v})/c^2}{\sqrt{1-\mathbf{v}^2/c^2}}$$

$$\sigma_v \equiv (1-1/\gamma_v)(c^2/\mathbf{v}^2) = \frac{1}{1+1/\gamma_v}$$

The following equations show the relationship between the space-contraction term σ_v ($0.5 \le \sigma_v \le 1.0$) & the time-dilation term δ_{uv} ($0 < \delta_{uv}$) in the SR variable calculations (ref [4]).

§1.2 Clock Rate (*dt*) & Rest Time (*dτ*) Transformations

$$\frac{dt'}{dt} = \delta_{uv} \qquad \frac{dt}{d\tau} = \gamma_v$$

§1.3 Mass Derivative (*dm*) Transformation from Rest Time (*τ*)

$$\frac{dm}{dt} = \frac{1}{\gamma_v}\frac{dm}{d\tau}$$

§2 4-Vector Equations (ref [5])

§2.1 4-Vector Distance & Time (***r***, *ct*) Lorentz Transformations

$$\mathbf{r}' = \mathbf{r} + \gamma_v \mathbf{v}\left[\frac{(\mathbf{r}\cdot\mathbf{v})}{c^2}\sigma_v - t\right] \qquad t' = \gamma_v\left[t - \frac{(\mathbf{r}\cdot\mathbf{v})}{c^2}\right]$$

$$(\mathbf{r}')^2 = \mathbf{r}^2 + c^2\left[(t')^2 - t^2\right]$$

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§2.2 4-Vector Momentum & Energy (***p***, *E/c)* Definitions & Transformations

$$\mathbf{p} \equiv \gamma_u m \mathbf{u} \qquad E \equiv \gamma_u m c^2$$

$$\mathbf{p}' = \mathbf{p} + \frac{\gamma_v \mathbf{v}}{c^2}\left[(\mathbf{p}\cdot\mathbf{v})\sigma_v - E\right] \qquad E' = \gamma_v\left[E - (\mathbf{p}\cdot\mathbf{v})\right]$$

$$(\mathbf{p}')^2 = \mathbf{p}^2 + \frac{1}{c^2}\left[(E')^2 - E^2\right]$$

§2.3 4-Vector Force & Power (***F***, *P/c*) Definitions & Transformations (ref [10])

$$\mathbf{F} \equiv \frac{d\mathbf{p}}{d\tau} = \gamma_u \frac{d\mathbf{p}}{dt} \qquad P \equiv \frac{dE}{d\tau} = \gamma_u \frac{dE}{dt}$$

$$\mathbf{F}' = \mathbf{F} + \gamma_v \frac{\mathbf{v}}{c^2}\left[(\mathbf{F}\cdot\mathbf{v})\sigma_v - P\right] \qquad P' = \gamma_v\left[P - (\mathbf{F}\cdot\mathbf{v})\right]$$

$$(\mathbf{F}')^2 = \mathbf{F}^2 + \frac{1}{c^2}\left[(P')^2 - P^2\right]$$

§2.4 4-Vector Variable Mass ($\dot{m}\mathbf{w}_0$, $\dot{m}c^2$) Rocket Motor Definitions (may be errors)

$$\mathbf{a}_R = \frac{\dot{m}}{\gamma_u^2 m}\left[\mathbf{w}_0 - \mathbf{u}\frac{(\mathbf{u}\cdot\mathbf{w}_0)}{c^2}\sigma_u\right] \qquad \dot{m} \equiv \frac{dm}{d\tau}$$

$$\mathbf{F}_R = \dot{m}\left[\mathbf{w}_0 + \frac{\gamma_u \mathbf{u}}{c^2}\left((\mathbf{u}\cdot\mathbf{w}_0)\sigma_u - 1\right)\right]$$

§2.5 4-Vector Charge density & Current Density (***J***, *ρc*) Definitions & Transformations (ref [11])

To Be Determined

§3 3-Vector Equations

§3.1 Velocity ↔ Momentum (***u*** ↔ ***p***) Equivalence

$$\mathbf{u} = \frac{\mathbf{p}}{\sqrt{m^2 + (\mathbf{p}/c)^2}} \qquad \mathbf{p} \equiv \gamma_u m \mathbf{u} = \frac{m\mathbf{u}}{\sqrt{1 - (\mathbf{u}/c)^2}}$$

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§3.2 Velocity ($\boldsymbol{u}$) Transformation

$$\boldsymbol{u}' = \frac{1}{\delta_{uv}}\left[\boldsymbol{u}+\gamma_v \boldsymbol{v}\left[\frac{(\boldsymbol{u}\cdot\boldsymbol{v})}{c^2}\sigma_v - 1\right]\right]$$

$$\left(\frac{\boldsymbol{u}'}{c}\right)^2 = \left(1 - \frac{1}{(\delta_{uv}\gamma_u)^2}\right) \qquad \gamma_{u'} = \delta_{uv}\gamma_u$$

§3.3 Acceleration ($\boldsymbol{a}$) Transformation

$$\boldsymbol{a}' = \frac{1}{\delta_{uv}^2}\left[\boldsymbol{a} - \frac{\gamma_v(\boldsymbol{a}\cdot\boldsymbol{v})}{\delta_{uv}c^2}(\boldsymbol{v}\sigma_v - \boldsymbol{u})\right]$$

$$\boldsymbol{a}'^2 = \frac{1}{\delta_{uv}^4}\left[\boldsymbol{a}^2 + \frac{\gamma_v(\boldsymbol{a}\cdot\boldsymbol{v})}{\delta_{uv}c^2}\left[2(\boldsymbol{a}\cdot\boldsymbol{u}) - \frac{\gamma_v(\boldsymbol{a}\cdot\boldsymbol{v})}{\delta_{uv}\gamma_u^2}\right]\right]$$

§3.4 4-Vector Force & Power ($\boldsymbol{F}$, P/c) time derivatives ↔ Acceleration, rest mass & rest mass rate of change ($\boldsymbol{F}$, P/c ↔ $\boldsymbol{a}$, m, $dm/d\tau$) Equivalence (ref [10])

$$\boldsymbol{F} = \gamma_u^2 m\,\boldsymbol{a} + \boldsymbol{u}\left(\gamma_u^4 m\frac{(\boldsymbol{a}\cdot\boldsymbol{u})}{c^2} + \gamma_u\frac{dm}{d\tau}\right) \qquad P = \gamma_u m c^2\left[\gamma_u^3\frac{(\boldsymbol{a}\cdot\boldsymbol{u})}{c^2} + \frac{1}{m}\frac{dm}{d\tau}\right]$$

$$\boldsymbol{a} = \frac{1}{\gamma_u^2 m}\left[\boldsymbol{F} - \boldsymbol{u}\left(\frac{(\boldsymbol{F}\cdot\boldsymbol{u})}{c^2} + \frac{1}{\gamma_u}\frac{dm}{d\tau}\right)\right] \qquad \boldsymbol{a} = \frac{1}{\gamma_u^2 m}\left[\boldsymbol{F} - \boldsymbol{u}\frac{P}{c^2}\right]$$

§3.5 Distance Difference ($\boldsymbol{\Delta r}$) Transformation with constant target velocity ($\mathbf{u}_t$)

$$\Delta\boldsymbol{r}' = \Delta\boldsymbol{r} - \frac{\gamma_v(\Delta\boldsymbol{r}\cdot\boldsymbol{v})}{\delta_{uv}c^2}(\boldsymbol{v}\sigma_v - \boldsymbol{u}_t)$$

$$(\Delta\boldsymbol{r}')^2 = \Delta\boldsymbol{r}^2 + \frac{\gamma_v(\Delta\boldsymbol{r}\cdot\boldsymbol{v})}{\delta_{uv}c^2}\left[2(\Delta\boldsymbol{r}\cdot\boldsymbol{u}_t) - \frac{\gamma_v(\Delta\boldsymbol{r}\cdot\boldsymbol{v})}{\delta_{uv}\gamma_u^2}\right]$$

§4 Poynting Vector Equations

§4.1 Electromagnetic Fields ($\boldsymbol{E}$ & $\boldsymbol{B}$) from Linear Trajectory Source Charge in MKSA

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unit (may be errors)

$$\boldsymbol{E_s} = \frac{1}{4\pi\epsilon_0}\left[\frac{\gamma_s q_s \Delta\boldsymbol{r_{st}}}{[(\Delta\boldsymbol{r_{st}})^2+(\gamma_s(\Delta\boldsymbol{r_{st}}\cdot\boldsymbol{u_s})/c)^2]^{3/2}}\right]$$

$$\boldsymbol{B_s} = \frac{1}{c^2}\boldsymbol{u_s}\,\boldsymbol{x}\,\boldsymbol{E_s}$$

$$\boldsymbol{F_t} = q_t[\boldsymbol{E_s}+(\boldsymbol{u_t}\,\boldsymbol{x}\,\boldsymbol{B_s})]$$

§4.2 Electromagnetic Field (***E*** & ***B***) Transformations in MKSA units:(may be errors)

$$\boldsymbol{E_s}' = \gamma_v[\boldsymbol{E_s}+(\boldsymbol{v}\,\boldsymbol{x}\,\boldsymbol{B_s})-\sigma_v\boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{E_s})/c^2]$$

$$\boldsymbol{B_s}' = \gamma_v[\boldsymbol{B_s}-(\boldsymbol{v}\,\boldsymbol{x}\,\boldsymbol{E_s})/c^2-\sigma_v\boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{B_s})/c^2]$$

§4.3 Plane Wave Poynting Vector (***S***) Transformation MKSA units:(may be errors)

$$S' = \frac{1}{\mu_0}\boldsymbol{E'_s}\,\boldsymbol{x}\,\boldsymbol{B'_s} \qquad 'S = \frac{1}{\mu_0}|\boldsymbol{E'_s}||\boldsymbol{B'_s}|$$

§4.4 Electromagnetic Wavelength (λ) Doppler Transformation

$$\lambda'_{st} = \gamma_u[1-(\hat{\boldsymbol{n}}_{\boldsymbol{st}}\cdot\boldsymbol{u_{st}})/c]\lambda_{st}$$

§5 Electromagnetic Field Tensor Equations in MKSA units: (ref [7]) (may be errors)

§5.1 Auxiliary variable equation

$$\boldsymbol{\beta}_v \equiv \frac{\boldsymbol{v}}{c}$$

§5.2 Lorentz Transformation in symmetric vector form for transformation velocity ***v***

$$\begin{pmatrix} r'_{ct} \\ r'_x \\ r'_y \\ r'_z \end{pmatrix} = [L_v]\begin{pmatrix} r_{ct} \\ r_x \\ r_y \\ r_z \end{pmatrix} \qquad L_v = \begin{bmatrix} \gamma_v & -\gamma_v\beta_x & -\gamma_v\beta_y & -\gamma_v\beta_z \\ -\gamma_v\beta_x & 1+\gamma_v\sigma_v\beta_x^2 & \gamma_v\beta_x\beta_y & \gamma_v\beta_x\beta_z \\ -\gamma_v\beta_y & \gamma_v\beta_x\beta_y & 1+\gamma_v\sigma_v\beta_y^2 & \gamma_v\beta_y\beta_z \\ -\gamma_v\beta_z & \gamma_v\beta_x\beta_z & \gamma_v\beta_y\beta_z & 1+\gamma_v\sigma_v\beta_z^2 \end{bmatrix}$$

§5.3 Maxwell's Equations in EM field tensor ($F^{\beta\gamma}$) form

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$$\partial^\alpha F_D^{\beta\gamma} - \partial^\beta F_D^{\alpha\gamma} = \mu_0 J^\gamma \qquad \partial^\alpha F_I^{\beta\gamma} - \partial^\beta F_I^{\alpha\gamma} = 0$$

§5.4 The 4-vector potential (A^α) described from local 4-vector current densities (J^α). Retarded time (t') is used.

$$A^\alpha(\mathbf{x},t) \equiv \frac{\mu_0}{4\pi}\int_V \frac{J^\alpha(\mathbf{x}',t')}{|\mathbf{x}-\mathbf{x}'|} d^3x' \qquad t' = t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}$$

§5.5 Direct (F_D) & Inverted (F_I) field strength tensors of ***E*** & ***B*** fields are given by the formulas

$$F_D^{\alpha\beta} \equiv \partial^\alpha A^\beta - \partial^\beta A^\alpha \qquad (E_i/c) \rightarrow B_i \quad B_j \rightarrow (-E_j/c)$$

$$F_D^{\alpha\beta} = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{bmatrix} \qquad F_I^{\alpha\beta} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & -E_z/c & 0 & E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{bmatrix}$$

§5.6 EM force on charge: Lorentz Law:

To be determined

E. Newton Physics from SR Vector Equations

All of Newton's equations are recovered from the previous cited SR equations when *c* the speed of light is taken to infinity ∞. ($1/c \rightarrow 0$)

"All time flows unchangingly at the same pace, in the same increments, from the infinite past into the infinite future."
- Isaac Newton, *Philosophiae Naturalis Principia Mathematica,* 1687

Einstein assumed the above Newtonian statement is incorrect. In Newtonian physics:

$$t' = t \qquad \frac{dt'}{dt} = 1$$

"The laws of physics are invariant with respect to Lorentz transformations."
- Albert Einstein, *Autobiographical Notes*, 1949

The Lorentz transformations are derived to maintain the measurement of light speed *c* as constant in all reference frames (ref [2]). From this document, the Lorentz transformation of time *t* is:

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$$t' = \gamma_v\left[t - \frac{(\boldsymbol{r}\cdot\boldsymbol{v})}{c^2}\right] \qquad \frac{dt'}{dt} = \gamma_v\left[1 - \frac{(\boldsymbol{u}\cdot\boldsymbol{v})}{c^2}\right] = \delta_{uv}$$

To recover Newtonian physics, when the speed of light *c* is taken to infinity ∞, the auxiliary variables in this document become:

$$\lim_{c\to\infty}\gamma_v = 1 \qquad \lim_{c\to\infty}\delta_{uv} = 1 \qquad \lim_{c\to\infty}\sigma_v = \frac{1}{2}$$

All of the transformations given previously, except for that of energy *E* & its derivative power *P,* revert to Newtonian form. Energy *E* is defined differently in Special Relativity. The Newtonian kinetic energy *KE* of a moving particle is given as (ref [6]):

$$KE = \frac{1}{2}mv^2$$

The *KE* equation can be recovered (ref [1]) by taking the Taylor Series Expansion (ref [8]) of γ_v around ***v**/c=0*:

$$\gamma_v \approx 1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 + \frac{3}{8}\left(\frac{v}{c}\right)^4 + \ldots \qquad E \equiv \gamma_v m c^2 \approx mc^2 + \frac{1}{2}mv^2 + \ldots$$

F. References

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G. Author's Note:

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Dahlgren Vector Equations of Special Relativity

I initiated collection / derivation of these equations in Dahlgren, Virginia in the late 1990's. I had been place on mental disability while my then wife still worked at a military Research & Development (R&D) facility there. So named "Dahlgren Vector Equations of Special Relativity" because, even though at an R&D facility with lots of formal education and lofty goals, unusual mannerism & ill-timed coughs can project one into the tabloids for decades.

In some cases, I was in error in recording the proper references. Hopefully, these equations can be useful in calculating numerical quantities in Special Relativity. Most of the equations have been subjected to validation through numerical coding (see chkSRel.cpp & chkSRel.txt in chkSRel.zip).

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