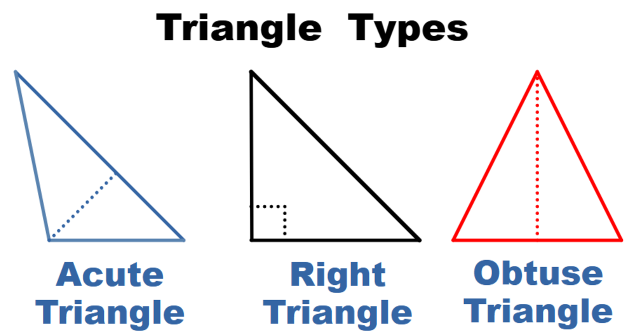
Trigonometry Demystified

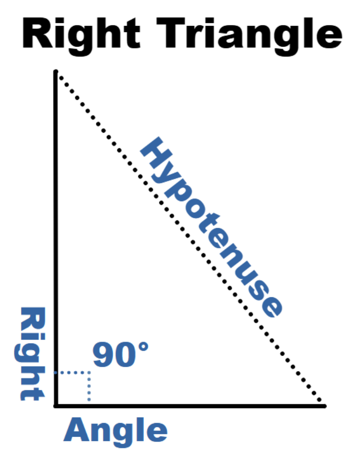
Objective

By request, this article will address Trigonometry to the extent that derivation of the Galileo's Artillery Equation is explained. Trigonometry is defined as the "Study of Triangles". How interesting can ***that*** be? You have to approach these things with a curious mind & discover "Why!" The knowledge society retains from as far back as ancient Greece is retained (ref [1]), because we find it useful today. Basic Algebra is helpful in understanding trigonometry, but illustrations of triangles are also necessary. Novices limit their study to "right" triangles. From the 3 types of triangles, acute, obtuse & right, layman divide all triangles into "right" triangles before their properties can be studied with the buttons on their calculators. The figure above, illustrates the 3 types of triangles. In the center right triangle, the "right angle" is also shown along with ways the obtuse & acute triangles could be bisected to form right triangles.



The Hypotenuse

It's very important to realize once their 3 angles are set, all triangles with those angles look the same. It's a matter of scale. So, trigonometry describes the properties of the "unit" triangle in depth & people with calculators scale that data up or down depending on their specific triangles. To define the unit triangle, we get to learn the term "hypotenuse" which only applies to right triangles (ref [2]). The hypotenuse is the triangle side always opposite of the right angle & is always assigned the length of unity, i.e., one "1" for triangle study. The trig buttons on a calculator tell the user the lengths of the other triangle sides when the hypotenuse of the unit triangle is one. Each other side of the unit triangle always measures between negative one and positive one.



Greek Letters – What Do They Mean?

The "sweep" in each triangle corner between a pair of the triangle sides is termed an "angle", measured in degrees, from 0° to 360° of arc. Angles can have any value, zero & negative included. A 720° value means one has gone in a circle counter-clockwise twice. A –900° value indicates a full 2½ rotation clockwise. The right angle of a right triangle is always 90° & the adjacent sides are termed "perpendicular".

In countries using the Latin alphabet, angles are sometimes denoted in mathematics using lower-case Greek letters. I guess we honor Pythagoras (c 570 BC – c 495 BC) (ref [3]), when we use the Greek letters, which are still used in the modern Greek language. Can mathematicians "spell" any Greek words with those Greek letters? Only if they take a course in Greek! Are mathematicians channeling any transcendental ideas of ancient Greek philosophers like Plato, Aristotle or Socrates with those Greek letters? Absolutely Not! That's way too much hyperbole!

Mathematicians have essentially run out of Latin letters. Most importantly, the Greek letters separate themselves from the Latin spelled trigonometric functions. The Greek character (θ) is the lower-case "theta" & is often used as an angle variable. In Trigonometry textbooks, a back-of-the-book appendix (see ref [4]) usually gives a one page list of the 24 upper & lower-case Greek letters with "English" pronunciation labels of each.

Word processors implement Unicode (refs [5] & [6] & [7]) inserting Greek letters as "special characters", or you can simply copy **<CTRL–C>** & paste **<CTRL–V>** from the chart to the right.

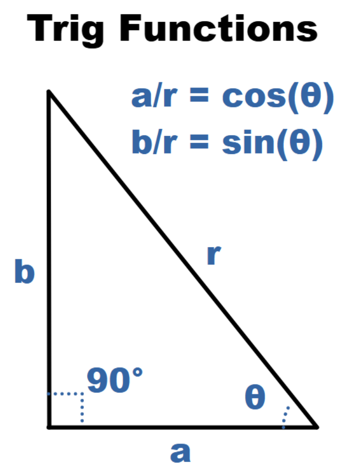
Greek Alphabet

English case upper	English case lower	English label
Α	α	Alpha
Β	β	Beta
Γ	γ	Gamma
Δ	δ	Delta
Ε	ε	Epsilon
Ζ	ζ	Zeta
Η	η	Eta
Θ	θ	Theta
Ι	ι	Iota
Κ	κ	Kappa
Λ	λ	Lambda
Μ	μ	Mu
Ν	ν	Nu
Ξ	ξ	Xi
Ο	ο	Omicron
Π	π	Pi
Ρ	ρ	Rho
Σ	σ	Sigma
Τ	τ	Tau
Υ	υ	Upsilon
Φ	φ	Phi
Χ	χ	Chi
Ψ	ψ	Psi
Ω	ω	Omega

A couple of Greek letters are avoided because they are difficult for Latin alphabet users to write, i.e., lower-case "zeta" (ζ) & "xi" (ξ). Upper / lower-case "omicron" (o) & "iota" (ι) are also never used in trigonometry, being indistinguishable from numerical zero & one, respectively. The lower-case Greek letter "pi" (π) is reserved in higher trigonometry as the ratio of a circle's circumference length with its diameter. Otherwise, choose your ***favorite*** lower-case Greek letter for an "angle variable" as you accomplish your trig problems.

By no means are Greek letters exclusively reserved for angles. Hendrik Lorentz (1853 – 1928) (ref [8]) chose the lower-case Greek letter "gamma" (γ), in his derivation of transformation equations in his "almost but not quite" contributions to Einstein's Special

Relativity (ref [9]). Oddly, the Lorentz term (*γ*) helps define a boost angle (*α*) (*cosh*(*α*) = *γv*). This angle describes coordinate rotation into the time-axis of spacetime as an object travels at speed (*v*), the speed ratio to light speed & close to unity (ref [10]). Another example, the lower-case Greek letter "mu" in (μm) abbreviation denotes the metric length microns (1 μm = 10^{-6} meters) (ref [11]).

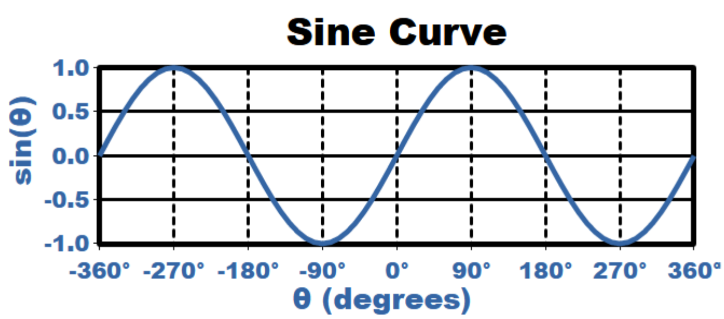


The Tale of Two Functions

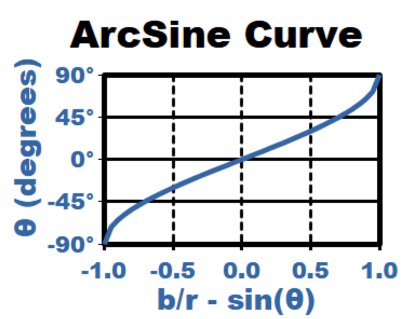
Triangles are defined by the length of their sides & the angles between their sides. A triangle side is measured in any applicable length, i.e., inches, meters, miles, kilometers, light years. Angles are usually expressed in degrees or radians, but you can learn about radians in a math course. Trigonometric functions relate the angles & sides of a triangle. Several functions are defined in trigonometry for a triangle, but for the novice with a calculator, only two are necessary. The functions are:

a/r = cos(θ) ("cos" is an abbreviation for "cosine" & pronounced "co-sign")
b/r = sin(θ) ("sin" is an abbreviation for "sine" & pronounced "sign")

The cosine function supplies the length of the adjacent side (*a*) of angle (*θ*) for the unit triangle (*r = 1*). The sine function supplies the length of the opposite side (*b*) of angle (*θ*) for a unit triangle (*r = 1*). If your triangle of study has a hypotenuse (*r*) different from unity, then the "sin" & "cos" buttons on your calculator gives the fractional values pre-divided by (*r*). To get the sides of your triangle with hypotenuse (*r*), multiply the calculator's answer by a factor (*r*).



The graph above is the proverbial "sine curve" plotted. The curve shows 2 cycles repeating every 360°. The curve oscillates from 0.0 @ -180° to -1.0 @ -90° to 0.0 @ 0° to +1.0 @ 90° returning to 0.0 @ 180°. The cosine function plot is almost identical, but shifted to the left 90°.



Inverse Trigonometry Functions

By request, inverse trig functions are outlined, here. On scientific calculators are inverse trig buttons which allows one to go the other way. Choose the displayed angle units as degrees (consult the calculator manual). Then, the inverse cosine function, of many names "$Cos^{-1}(x)$" or "*Acos*(*x*)" or "*Arccos*(*x*)", answers the question: "Given a fractional cosine value, what angle would produce that ratio?”

$$Acos(a/r) = \theta \qquad (0° \le \theta \le 180°)$$
$$Asin(b/r) = \theta \qquad (-90° \le \theta \le 90°)$$

From the graph above, the inverse sine function domain of fractional lengths (-1.0 to +1.0) corresponds to a range of angles (-90° to +90°) every 180° & repeats. Inverse trig functions limit the output angle. An angle range is chosen so that one angle value corresponds to each input fractional length. That’s a fundamental requirement of a function (ref [12]).

The Pythagorean Theorem

Pythagóras ho Sámios (c 570 BC – c 495 BC) contributed his equation for right triangles as his primary claim to fame (ref [3]). That equation is (ref [13]):

$$r = \pm\sqrt{a^2 + b^2} \quad ⇨ \quad r^2 = a^2 + b^2$$

For our trigonometric functions:

$$1 = \sqrt{\cos^2\theta + \sin^2\theta} \quad ⇨ \quad 1 = \cos^2\theta + \sin^2\theta$$

Many equations relate trigonometric angles to the lengths of any triangle side. A student can take many courses in mathematics & encounter Trigonometry, Pythagoras & his theorem several times over.

Hitting the Target With a Cannon Ball

To apply Trigonometry, we will take our ballistic cannon ball investigation of Galileo (1564 – 1642) (ref [14] & ref [15]). A 2-Dimensional Cartesian coordinate system can be used to plot the cannon ball trajectory. Most importantly, the cannon ball initial conditions, i.e., its muzzle velocity & cannon elevation angle will be incorporated in an equation to make sure the cannon ball trajectory ends on the desired target a distance (R_x) down range.

R Descartes 1596 – 1650

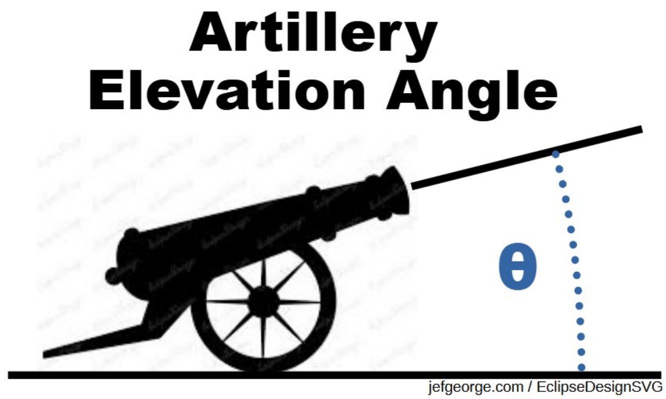
René Descartes (refs [16] & [17]) is credited with allowing 3-Dimensional space to be described by perpendicular axes (ref [18]). Monsieur Descartes was a Frenchman, so his name is pronounced “DAY-cart”. The Cartesian coordinate system’s name is derived from “Descartes”. Based on Descartes’ work, the initial cannon ball velocity (v_0) can be divided into vertical & horizontal components per the Pythagorean Theorem as:

$$(v_{0x})^2 + (v_{0y})^2 = (v_0)^2 \quad ⇨ \quad v_{0x} = v_0 \cos\theta \;\; \& \;\; v_{0y} = v_0 \sin\theta$$

In applying a 2-Dimensional Cartesian coordinate system, the cannon ball’s initial vertical speed (v_{0y}) & constant horizontal speed (v_{0x}) can be related to the cannon elevation angle (θ) measured from the cannon’s horizontal (ref [19]). Clearly, within the vertical plane of the cannon, the cannon ball’s muzzle speed squared (v_0^2) is invariant under the cannon’s elevation angle (θ).

Leonhard Euler (1707 – 1783) (ref [20]) devised 3 “Euler Angles” in 3-Dimensional Coordinate space that define any axes orientation. Then, length squared (r_0^2) of any 3D vector is invariant under any 3D spatial rotation. Herr Euler was a Swiss German & his name is pronounced as “OIL-er.”

Level terrain is assumed, then flight time of the cannon ball can be found from the measured range (R_x) & unknown horizontal speed ($v_0 \cos\theta$). In the horizontal x-dimension:



$$R_x = v_{0x}\, t_{flight} \quad \Rightarrow \quad t_{flight} = R_x / v_{0x}$$

In the vertical y-dimension for level terrain:

$$y_{flight} = v_{0y}\, t_{flight} - ½\, g\, (t_{flight})^2 = 0$$
$$v_{0y} = ½\, g\, t_{flight}$$

Eliminating projectile flight time (t_{flight}) from the vertical coordinate equation:

$$v_{0y} = ½\, g\, [R_x / v_{0x}]$$

Efforts of Galileo & Isaac Newton (1642 – 1726) (ref [14] & [15] & [21]) relate the target range (R_x), cannon ball muzzle velocity (v_0) & earth’s gravity (g). Applying trigonometry of Pythagoras & Descartes incorporates the cannon elevation angle (θ) from its horizontal:

$$½\, gR_x = v_{0x} v_{0y} = (v_0)^2 \cos\theta \sin\theta \quad \Rightarrow \quad ½\, (gR_x)/(v_0)^2 = \cos\theta \sin\theta$$

From a mathematical handbook (ref [22]), trigonometric function calculations can often be simplified. Using a “double-angle” trig identity,

$$\cos\theta \sin\theta = ½ \sin(2\theta)$$
$$½\, (gR_x)/(v_0)^2 = ½ \sin(2\theta) \quad \Rightarrow \quad \sin(2\theta) = (gR_x)/(v_0)^2$$

Also, the inverse sine function, a.k.a. “*Sin⁻¹*(*x*)” or “*Asin*(*x*)” or “*Arcsin*(*x*)”, answers the question: “Given a fractional sine value, what angle would produced that ratio?” Note the value evaluated within the inverse sine function in this application will never be negative. The final equation is:

$$\theta = ½\, Asin[\, gR_x / (v_0)^2] \qquad (0° \le \theta \le 45°)$$

The initial muzzle velocity (v_0) of the cannon ball is measured before the battle. On the battlefield, the range (R_x) *to target* is estimated, the cannon ball is loaded, the barrel angle (θ) is adjusted & the artillery has a better chance of hitting the target. For artillery, elevation angles are normally measured from the horizontal in degrees (ref [23]). You can begin your Science / Technology / Engineering / Math (STEM) journey into Calculus here (ref [15]).

If you have made it to the end of this article & followed along, a STEM degree may be in your future. With a STEM degree, you really ***can*** “leave the world better than you found it."

References

[1] Brooks, MJ, PhD, Science Focus, Humble Triangle Shapes Human History, 2022.
[2] Wikipedia.org, Hypotenuse, 2023.
[3] Wikipedia.org, Pythagoras, 2023.
[4] Businessballs.com, Greek Alphabet: Upper & Lower Case, 2020.
[5] UNICODE, The World Standard for Text & Emoji's, 2023.
[6] HotSymbol.com, Greek Letter Symbols, 2023.
[7] George, Everett, *Ideas Contributions*, Unicode Selector, 2023.
[8] Wikipedia.org, Hendrik Lorentz, 2023.
[9] Wikipedia.org, Lorentz factor, 2022.
[10] Hamilton, Andrew JS, PhD, U of Colorado, Spacetime Wheel & Boost Angle, 2022.
[11] Wikipedia.org, Mircron, 2023.
[12] Wikipedia.org, Mathematical Functions, 2023.
[13] Wikipedia.org, Pythagorean theorem, 2023.
[14] Wikipedia.org, Galileo Galilei, 2023.
[15] George, Everett, *Ideas Contributions*, Calculus Demystified, 2023.
[16] Wikipedia.org, René Descartes, 2023.
[17] Crease, RP, *Physics World*, In Praise of Descartes, 11 Aug 2016.
[18] Wikipedia.org, Descartes: Cartesian coordinate system, 2022.
[19] Halliday, David, et al, *Principles of Physics*, 11th Ed, Wiley, 2020.
[20] Wikipedia.org, Leonhard Euler, 2024.
[21] Wikipedia.org, Isaac Newton, 2023.
[22] Spiegel, MR, PhD, et al, *Schaum's Math Handbook*, 5th Ed, McGraw-Hill, 2017.
[23] Holzner, S, PhD, *Physics for Dummies*, Calculate Projectile Range, Wiley, 2016.

Note: Please donate $50/year if you find Wikipedia useful!!!

Everett George 21 July 2024