Calculus Demystified

Objective

If one knows basic Algebra, one can learn the essence of Calculus. The best way to learn Calculus, is most probably the way Isaac Newton created Calculus. Wanting to investigate the motion of the planets, Galileo (1564 – 1642) (ref [1]) studied trajectories on Earth & left Newton (1642 – 1726) (ref [2]) some hints. By 1687, Newton had used these hints to great effect to invent Calculus as outlined in *Principia* (ref [3]). He solved the millennia old mysteries of planetary motion specifically (ref [4]) & motion in general.

Dropping Balls

Galileo was curious about ballistic motions & through ramps, set about measuring just how fast an object starts to drop (ref [5]). He found a ball drops 1 unit of length the 1st second, then 3 units of length the 2nd second, and 5 units of length the 3rd second. If one plots that out, a squared behavior of time verse height relationship becomes apparent (ref [1]).

Dropping Object Data

unit of time	height change	Δ height change
1	1	–
2	4	3
3	9	5
4	16	7

The Parabolic Arc

From his experiments, Galileo hypothesized the ballistic trajectory of a cannon ball with no wind was most likely an inverted parabola. A parabola is given by the equations:

$$height(t) = y(t) = v_{0y}\, t - kt^2 \qquad\qquad range(t) = x(t) = v_{0x}\, t$$

The variable (k) is a constant from measurement whose form is yet to be determined. The trajectory is defined through time in flight (t), horizontal range (x) & vertical height (y). Initial speed of the cannon ball (v_0) is measured leaving the cannon's muzzle. The horizontal speed of the cannon ball (v_{0x}) was seen as essentially constant neglecting wind resistance. Astutely, Galileo reasoned there was minimal wind resistance around the far away orbiting planets (ref [6]). Indeed, the planets moved across the sky without slowing down, year after year.

Newton & His Derivative

Newton wanted to relate the change in height (y) of a cannon ball with the change in time (t) in a given dimension over an interval (h) in time. The trajectory of an object should give an indication of its velocity (ref [2]). For Galileo's parabola:

$$v_y(t) \approx \frac{\Delta y}{\Delta t} = \frac{y(t+h) - y(t)}{(t+h) - t} = \frac{\left(v_{0y}(t+h) - k(t+h)^2\right) - \left(v_{0y}t - kt^2\right)}{(t+h) - t}$$

$$v_y(t) \approx \frac{\Delta y}{\Delta t} = v_{0y}\frac{(t+h)-t}{(t+h)-t} - k\frac{(t^2+2ht+h^2)-t^2}{(t+h)-t}$$

$$v_y(t) \approx \frac{\Delta y}{\Delta t} = v_{0y} - k\frac{(2ht+h^2)}{h} = v_{0y} - k(2t+h)$$

Newton wanted to know the "instantaneous" velocity along the trajectory (not just an approximation), so he took the interval (*h*) to zero. Later (1817), a mathematical construct was developed called a *limit* (ref [7]). In a *limit*, an expression is evaluated with (*h*) approaching zero, but (*h*) never reaching zero. From Newton:

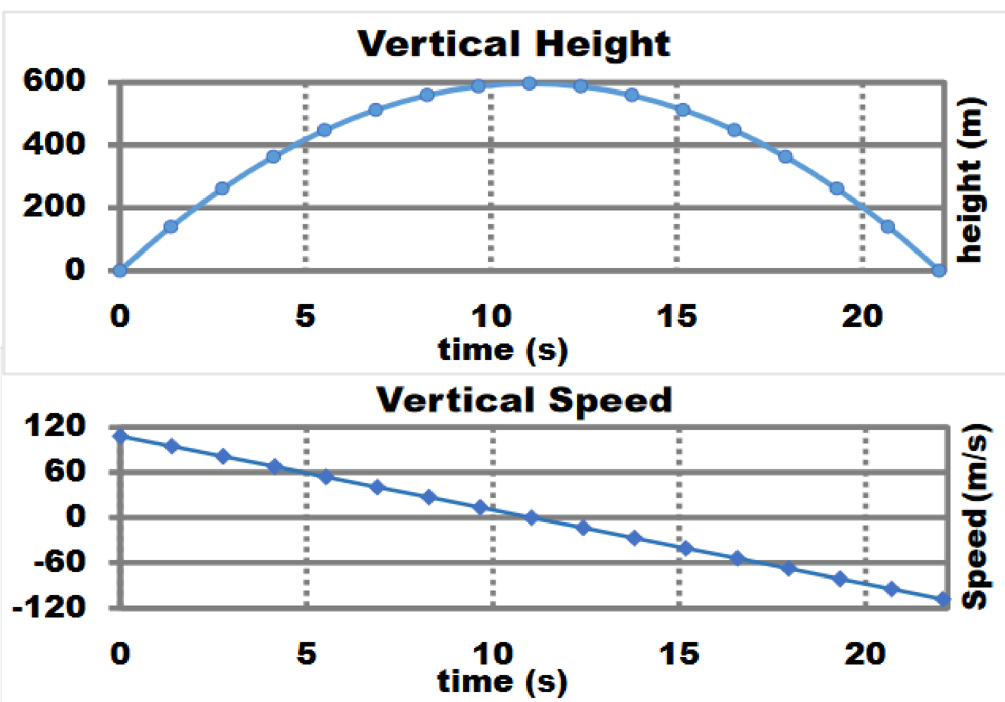
$y(t) = v_{0y}\, t - kt^2$ ⇨ with its derivative ⇨ $v_y(t) = v_{0y} - 2kt$

The force of gravity is averaged at the surface of the mid-latitudes of earth @ g = 9.8 m/s^2 (32.17 ft/s^2) (ref [8]). The value decreases with distance from the earth's center & latitude towards its equator. However, this constant value describes the acceleration of a falling object as Galileo observed it to a good approximation (ref [9]). Then the constant (*k*) is evaluated as:

$k = g/2$ ⇨ $v_y(t) = v_{0y} - gt$ ⇨ with its integral ⇨ $y(t) = v_{0y}\, t - gt^2/2$

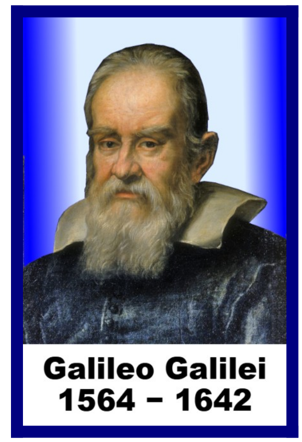
BTW, the term "g-force" is derived from this "*g*" constant. The measure is the "acceleration that causes a perception of weight, with a g-force of 1 g … equal to the conventional value of gravitational acceleration on Earth." G-force is expressed in multiples of 9.8 m/s^2 acceleration (ref [10]).

That's the Math – Here's the Graph



To understand Calculus, one has to understand the graphs to the left. The top plot is the parabolic curve ($y(t)$) of a cannon ball arcing across the sky, plotted against time in flight. Because the horizontal speed (v_{0x}) of the cannon ball is assumed constant, the downrange measure reflects time in flight (t). The bottom plot is vertical velocity ($v_y(t)$), the derivative of the top plot location ($y(t)$), and gives the “instantaneous” vertical speed (v_y) of the cannon ball throughout its trajectory.

The cannon ball is fastest at about 120 m/s vertically out of the cannon muzzle & just before hitting the ground. This example is based on a weak muzzle velocity of 213 m/s (699 ft/s) or about 767 kph (476 mph) @ a barrel elevation of about 30 degrees (ref [11]). The upward / positive vertical speed decreases until @ about 11 seconds after firing, the vertical velocity is zero, changes sign & direction. At that time, the linear vertical velocity plot crosses the horizontal axis & zero value, then goes negative back toward Earth. At 11 seconds after firing, the cannon ball also reaches its highest. The negative (downward) slope of the vertical velocity plot is the gravitational acceleration, i.e., 9.8 meter per second per second (m/s^2).



Hitting the Target

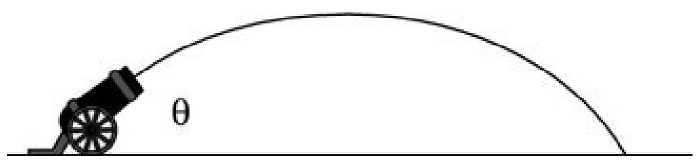
Galileo had to know Trigonometry beyond Algebra to help gunners of the time hit a target with a cannon ball, but below is the equation for getting the correct cannon elevation angle to hit the downrange target (refs [12] & [13] & [14]). BTW, Science / Technology / Engineering / Math (STEM) majors sometimes have to conquer Trigonometry as well as Calculus.

$$R_x = [(v_0)^2/g] \cdot \sin(2\theta)$$

Note the value evaluated within the inverse sine function will never be negative. The final equation is:

$$\theta = \tfrac{1}{2}\,\mathrm{Asin}[(gR_x)/(v_0)^2] \qquad (0° \le \theta \le 45°)$$

The initial muzzle velocity (v_0) of the cannon ball is measured before the battle. On the battlefield, the range (R_x) to target is estimated, the cannon ball is loaded, the barrel elevation angle (θ) is adjusted, and the artillery has a better chance of hitting the target. For artillery, elevation angles are normally measured from the horizontal in degrees. You can begin your STEM journey into Trigonometry here (ref [14]).



Calculus Notations

The Calculus notation (ref [15]) is unique, but simple enough. Remember, integration is “anti-differentiation”, the opposite of differentiation. From justifications similar to the previous “h” proofs given herein & with qualifications, the derivative & integral (respectively) of an independent variable (t) raised to the exponent (n) is:

$$n\,t^{n-1} = \frac{d}{dt}t^n \quad (n \neq 0) \qquad\qquad \frac{t^{n+1}}{n+1} = \int t^n\,dt \quad (n \neq -1)$$

For our ballistic example, (*n=1* & *n=2*) to determine the derivative of height ($y(t)$), and (*n=0* & *n=1*) to determine the integral of vertical velocity ($v_{0y}(t)$), in this application of Calculus. Remember, a variable raised to the power of zero (t^0) is defined as unity (*1*) & disregarded in multiplication.

Mathematical Rigor

As requested, Math will be discussed based on its method of validation. In the abbreviation STEM, Science / Technology / Engineering / Math, the 1st three fields incorporate Science, whereas the last field of Mathematics operates in a different realm. Science applies the Scientific Method (ref [16]). “The scientific method involves making conjectures ... deriving predictions ... carrying out ... empirical observations based on those predictions … to determine whether observations agree with or conflict with the expectations.”

The Laws of Science are assumed constant across time & can ***always*** be retested (ref [5]). The major religions (Hinduism, Judaism, Christianity, Islam) involve events that occurred long ago among ancient people that, for the most part, can not be tested by Science. Scientific conclusions cannot be drawn & supporting events of these religions must be accepted on faith (ref [17]).

Mathematicians, for the most part, cannot test their conclusions in the real world, either. The existence of a perfect circle in the real world is debatable (ref [18]). As a result, Mathematics relies on “rigor” (ref [19]). To prove an idea, a rigorous proof is given “where all assumptions need to be stated and nothing can be left implicit.” For example, if a computer program uses Math that divides a number by the variable “*x*”, the programmer has assumed that “*x*” cannot be zero. If an important computer was functioning when it divided by zero, for the most part, it will stop its important execution, unless the undefined division was anticipated.

I am not a Mathematician & did ***not*** attempt to present Calculus under rigor in this article. However, I value the efforts that Mathematicians put forth to keep Mathematics accurate. At the college level, Math is usually taught to all students by Mathematicians from the viewpoint of rigor. In addition, many unsolved problems in Science & Engineering could be solved, if only they were brought to the attention of a Mathematician. When one chooses a Mathematics career out of STEM, it’s slightly different from the other fields, but still much appreciated!

If you have made it to the end of this Math excursion & followed along, a STEM degree may be in your future. With a STEM degree, you really ***can*** “leave the world better than you found it.”

References

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